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Approximations for certain hyperbolic functions by partial sums of their Taylor series and completely monotonic functions related to gamma function

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ABSTRACT

In this paper, we establish some lower and upper bounds for certain hyperbolic functions in terms of partial sums of their Taylor series. These allow us to present two completely monotonic functions involving gamma function. As consequences, sharp Burnside type approximations for gamma function, sharp Detemple–Wang type approximations for psi function, and sharp bounds for polygamma functions are deduced, which generalize and improve some known results.

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1. Introduction

For $\operatorname{Re}(z) > 0$ the classical Euler's gamma function Γ and psi (digamma) function ψ are defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \qquad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{1.1}$$

respectively. The derivatives ψ', ψ'', ψ''', ... are known as polygamma functions. For $\ln \Gamma(z)$, Binet has established the first formula

$$\ln \Gamma(z) = \left(z - \frac{1}{2}\right) \ln z - z + \frac{1}{2} \ln(2\pi) + \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) \frac{e^{-zt}}{t} dt, \quad \operatorname{Re}(z) > 0 \tag{1.2}$$

(see [17, p. 21, Eq. (5)]).

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Bernoulli polynomials $B_k(x)$ and Euler polynomials $E_k(x)$ are defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n, \qquad |t|<2\pi, \tag{1.3}$$

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}\frac{E_n(x)}{n!}t^n, \qquad |t|<\pi, \tag{1.4}$$

respectively. The Bernoulli numbers B_n are denoted by $B_n=B_n(0)$, while the Euler numbers are defined by $E_k=2^kE_k(1/2)$. It is known that for $n\in\mathbb{N}$,

$$B_{2n+1}=0 \text{ and } B_{2n}=(-1)^{n+1}|B_{2n}|, \tag{1.5}$$

$$E_{2n+1}=0 \text{ and } E_{2n}=(-1)^{n}|E_{2n}|. \tag{1.6}$$

And, the first few nonzero values are

$$\begin{array}{lllll} B_0=1, & B_1=-\frac{1}{2}, & B_2=\frac{1}{6}, & B_4=-\frac{1}{30}, & B_6=\frac{1}{42}, \\ E_0=1, & & E_2=-1, & E_4=5, & E_6=-61 \end{array}$$

(see [1, p. 804, Chapter 23]).

By the exponential generating functions (1.3) and (1.4) it is easy to deduce the following Taylor series expansions with radii of convergence R of hyperbolic functions [1, p. 804, Eq. (4.5.64), (4.5.65), (4.5.67)]:

$$\coth t=\sum_{k=0}^{\infty}\frac{2^{2k}B_{2k}}{(2k)!}t^{2k-1} \qquad (R=\pi), \tag{1.7}$$

$$\frac{1}{\sinh t}=-\sum_{k=0}^{\infty}\frac{2\left(2^{2k-1}-1\right)B_{2k}}{(2k)!}t^{2k-1} \qquad (R=\pi), \tag{1.8}$$

$$\tanh t=\sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{(2k)!}t^{2k-1} \qquad \left(R=\frac{\pi}{2}\right), \tag{1.9}$$

$$\frac{1}{\cosh t}=\sum_{k=0}^{\infty}\frac{E_{2k}}{(2k)!}t^{2k} \qquad \left(R=\frac{\pi}{2}\right). \tag{1.10}$$

In 1975, Slavić [35] proposed an integral representation of the function $x\mapsto\Gamma(x+1)/\Gamma(x+1/2)$, that is,

$$\frac{\Gamma(x+1)}{\Gamma(x+1/2)}=\sqrt{x}\exp\left[\sum_{k=1}^{n}\frac{\left(1-2^{-2k}\right)B_{2k}}{k(2k-1)x^{2k-1}}+\int_0^\infty\left(\frac{\tanh t}{2t}-\sum_{k=1}^{n}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2(2k)!}t^{2k-2}\right)e^{-4tx}dt\right]. \tag{1.11}$$

Then he claimed that "Since the sign of the sub-integral value and the sign for $(-1)^n$ are equal, the following inequality is valid

$$\sqrt{x}\exp\sum_{k=1}^{2m}\frac{\left(1-2^{-2k}\right)B_{2k}}{k(2k-1)x^{2k-1}}<\frac{\Gamma(x+1)}{\Gamma(x+1/2)}<\sqrt{x}\exp\sum_{k=1}^{2l-1}\frac{\left(1-2^{-2k}\right)B_{2k}}{k(2k-1)x^{2k-1}}, \tag{1.12}$$

where m and l are natural numbers and $x>0$". Slavić's result has been cited or mentioned by several papers, such as [2,3,19,4,28].

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