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Algebraic structures in the sets of surjective functions

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ABSTRACT

In the paper we construct several algebraic structures (vector spaces, algebras and free algebras) inside sets of different types of surjective functions. Among many results we prove that: the set of everywhere but not strongly everywhere surjective complex functions is strongly $\mathfrak{c}$-algebrable and that its $2^{\mathfrak{c}}$-algebrability is consistent with ZFC; under CH the set of everywhere surjective complex functions which are Sierpiński–Zygmund in the sense of continuous but not Borel functions is strongly $\mathfrak{c}$-algebrable; the set of Jones complex functions is strongly $2^{\mathfrak{c}}$-algebrable.

1. Introduction

For some time now, many mathematicians have been looking at the largeness of some sets by constructing algebraic structures inside them. This approach is called *algebrability*. A comprehensive description of this concept as well as numerous examples and some general techniques can be found in the surveys [11,15].

Following R. Aron, A. Bartoszewicz, S. Głąb, V. Gurariy, D. Pérez-García, J.B. Seoane-Sepúlveda, [4–6, 12] let us recall the following notions:

Definition 1.1. Let κ be a cardinal number.

(1) Let $\mathcal{L}$ be a vector space and $A \subseteq \mathcal{L}$. We say that A is *κ-lineable* if $A \cup \{0\}$ contains a κ-dimensional subspace of $\mathcal{L}$;
(2) Let $\mathcal{L}$ be a commutative algebra and $A \subseteq \mathcal{L}$. We say that A is *κ-algebrable* if $A \cup \{0\}$ contains a κ-generated subalgebra B of $\mathcal{L}$ (i.e. the minimal cardinality of the system of generators of B is κ).

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(3) Let $\mathcal{L}$ be a commutative algebra and $A \subseteq \mathcal{L}$. We say that A is *strongly κ-algebrable* if $A \cup \{0\}$ contains a κ-generated subalgebra B that is isomorphic to a free algebra.

Fact 1.2. *Observe that the set $X = \{x_\alpha : \alpha < \kappa\}$ is the set of free generators of some free algebra if and only if the set $\tilde{X}$ of elements of the form $x_{\alpha_1}^{k_1} x_{\alpha_2}^{k_2} \cdots x_{\alpha_n}^{k_n}$ is linearly independent; equivalently for any $k \in \mathbb{N}$, any nonzero polynomial P in k variables without a constant term and any distinct $x_{\alpha_1}, \ldots, x_{\alpha_k} \in X$, we have that $P(x_{\alpha_1}, \ldots, x_{\alpha_k})$ is nonzero.*

This paper is devoted to the investigation of algebrability properties of several classes of surjective functions. In the sequel we take into our considerations the following ones: *everywhere surjective type functions* (section 3), *Sierpiński–Zygmund functions* (section 4) and *Jones functions* (section 5). In the paper the symbol $\mathbb{K}$ stands for the set $\mathbb{R}$ or $\mathbb{C}$. We use a standard set theoretical notion. In particular, we identify ordinal number α with the set of all ordinals $\beta < \alpha$. Cardinal numbers are those ordinals α which are not equipotent with any $\beta < \alpha$. A cardinal number κ is called regular, if it cannot be decomposed into less than κ sets of cardinality less than κ. Moreover, to indicate the difference between the sets of natural numbers with or without 0 we use standard notation, i.e. $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\omega = \{0, 1, 2, \ldots\}$ (it should be mentioned here that ω is also identified with the first infinite cardinal).

2. The general method

We start with the simple, but in the view of further results, useful observation. It is a foundation of a powerful method whose particular case is the so-called *exponential like function method*.

Theorem 2.1. *Let κ be a cardinal number, $\mathcal{A} \subseteq \mathbb{K}^{\mathbb{K}}$ be a κ-generated algebra (resp. free algebra, vector space) and $\mathcal{G} \subseteq \mathbb{K}^{\mathbb{K}}$. Assume that there exists a function $F : \mathbb{K} \to \mathbb{K}$ such that $f \circ F \in \mathcal{G} \setminus \{0\}$ for every $f \in \mathcal{A} \setminus \{0\}$. Then $\mathcal{G}$ is κ-algebrable (resp. strongly κ-algebrable, κ-lineable).*

Proof. Observe that a function $h : \mathcal{A} \to \mathcal{G}$ defined by $h(f) = f \circ F$ is a morphism of structures. Hence, if $\mathcal{A}$ is κ-generated algebra (resp. free algebra or vector space), then $h[\mathcal{A}]$ has the same property. □

It turns out that algebra $\mathcal{A}$, which is very useful in several cases, is the $\mathfrak{c}$-generated free algebra of the so-called exponential like functions. In 2013, M. Balcerzak et al. (see [7]) introduced the following notion.

Definition 2.2. (See [7].) We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *exponential like* (of rank $m \in \mathbb{N}$), whenever for $x \in \mathbb{R}$

$$f(x) = \sum_{i=1}^{m} a_i e^{\beta_i x},$$

for some distinct nonzero real numbers $\beta_1, \ldots, \beta_m$ and some nonzero real numbers $a_1, \ldots, a_m$ (let us denote the set of all exponential like functions $f : \mathbb{R} \to \mathbb{R}$ by $\mathcal{EXP}(\mathbb{R})$).

In this setting, actually using the fact that $\mathcal{EXP}(\mathbb{R})$ is strongly $\mathfrak{c}$-algebrable, they proved the following.

Theorem 2.3. *(See [7].) Let X be a nonempty set and $\mathcal{G} \subseteq \mathbb{R}^X$. Assume that there exists a function $F : X \to \mathbb{R}$ such that $f \circ F \in \mathcal{G} \setminus \{0\}$ for every $f \in \mathcal{EXP}(\mathbb{R})$. Then $\mathcal{G}$ is strongly $\mathfrak{c}$-algebrable.*

It is a simple observation, looking at the proof of Theorem 2.3, that this result is a particular case of Theorem 2.1. Many applications of Theorem 2.3 can be found in the paper by A. Bartoszewicz et al. [9].