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Approximation and existence of Schauder bases in Müntz spaces of L_1 functions ☆

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ABSTRACT

An approximation of functions in Müntz spaces M_{Λ,L_1} of L_1 functions is studied with the help of Fourier series. Müntz spaces are considered with the Müntz condition and the gap condition imposed. It is proved that up to an isomorphism such spaces are contained in Weil–Nagy's class. Moreover, existence of Schauder bases in Müntz spaces M_{Λ,L_1} is investigated.

1. Introduction

One of the focus areas of functional analysis is devoted to the study of topological and geometric properties of topological vector spaces (see, for example, [9,12,13,19]). In particular, the study of bases in Banach spaces plays an important role (see, for example, [9,11,14–18,27] and the references therein). Moreover, for specific classes of Banach spaces many open problems remain. Among them Müntz spaces M_{Λ,L_1} play very important role (see [2–5,8,23] and the references therein). The latter spaces are defined as completions of linear spans over $\mathbf{R}$ or $\mathbf{C}$ of monomials t^λ with $\lambda \in \Lambda$ on the segment $[0,1]$ relative to the L_1 norm, where $\Lambda \subset [0,\infty)$, $t \in [0,1]$. As it is known in 1885 K. Weierstrass proved his theorem about polynomial approximations of continuous functions on the unit segment. But the space of continuous functions also possesses the algebraic structure. Later on in 1914 C. Müntz studied more general cases so that his spaces had not such algebraic structure. It was the problem whether they have bases. There was a progress for lacunary Müntz spaces

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satisfying the condition $\underline{\lim}_{n\to\infty}\lambda_{n+1}/\lambda_n > 1$ with a countable set Λ, but in general this problem was unsolved [8]. It is worth to mention that the monomials t^λ generally do not form a basis in the Müntz space.

In this paper results of investigations of the author on this problem are presented.

In section 2 an approximation of functions in Müntz spaces M_{Λ,L_1} of L_1 functions is studied with the help of Fourier series. Müntz spaces are considered with the Müntz condition and the gap condition imposed. It is proved that up to an isomorphism such spaces are contained in Weil–Nagy's class. For getting this result auxiliary Lemmas 3, 4 and Theorem 5 are proved. They permit to make reduction to a subclass of Müntz spaces M_{Λ,L_1} up to an isomorphism of Banach spaces so that a set Λ is contained in the set of natural numbers $\mathbf{N}$. Moreover, existence of Schauder bases in Müntz spaces M_{Λ,L_1} is investigated. The theorem about existence of Schauder bases in Müntz spaces M_{Λ,L_1} under the Müntz condition and the gap condition is proved.

All main results of this paper are obtained for the first time. They can be used for further studies of function approximations and geometry of Banach spaces. It is important for progress of mathematical analysis and also in different applications including measure theory and stochastic processes in Banach spaces.

2. Approximation and bases

To avoid misunderstanding we first present our notation and definitions.

1. Notation. Let $C([a,b],\mathbf{F})$ denote the Banach space of all continuous functions $f:[a,b]\to\mathbf{F}$ supplied with the absolute maximum norm

$$\|f\|_C := \max\{|f(x)| : x\in[a,b]\},$$

where $-\infty<a<b<\infty$, $\mathbf{F}$ is either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$.

Then $L_p((a,b),\mathbf{F})$ denotes the Banach space of all Lebesgue measurable functions $f:(a,b)\to\mathbf{F}$ possessing a norm as defined by the Lebesgue integral:

$$\|f\|_{L_p((a,b),\mathbf{F})} := (\int_a^b |f(x)|^p dx)^{1/p} < \infty,$$

where $1\le p<\infty$ is a fixed number, and $-\infty\le a<b\le\infty$.

Suppose that $Q=(q_{n,k})$ is a lower triangular infinite matrix with matrix elements $q_{n,k}$ having values in the real field $\mathbf{R}$ and satisfying the restrictions: $q_{n,k}=0$ for each $k>n$, where k, n are nonnegative integers. To each 1-periodic function $f:\mathbf{R}\to\mathbf{R}$ in the space $L_p((\alpha,\alpha+1),\mathbf{F})$ or in $C_0([\alpha,\alpha+1],\mathbf{F}) := \{f : f\in C([\alpha,\alpha+1],\mathbf{F}), f(\alpha)=f(\alpha+1)\}$ is counterposed a trigonometric polynomial

$$(1)\quad U_n(f,x,Q) := \frac{a_0}{2}q_{n,0} + \sum_{k=1}^n q_{n,k}(a_k\cos(2\pi kx) + b_k\sin(2\pi kx)),$$

where $a_k=a_k(f)$ and $b=b_k(f)$ are the Fourier coefficients of a function $f(x)$.

For measurable 1-periodic functions h and g their convolution is defined whenever it exists:

$$(2)\quad (h*g)(x) := 2\int_a^{a+1} h(x-t)g(t)dt.$$

Putting the kernel of the operator U_n to be: