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An existence result for a new class of impulsive functional differential equations with delay

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ABSTRACT

We prove the existence of bounded solutions of a new class of retarded functional differential equations with non-instantaneous impulses and delay on an unbounded domain. An application example is also included.

1. Introduction

The literature regarding impulsive differential equations is wide and rich; this is due to the intrinsic difficulties and the variety of the problems (see e.g. [1,9,13,17] and references therein), and also to the impressive amount of applications in different branches of Science. Indeed, impulsive differential equations naturally arise from several frameworks like pulse vaccination strategy (see [12,22]) or processes of fed-batch fermentation as in [14,25]. For a general introduction to the theoretical approaches and for more applications, we refer to the monographs [3,4,20].

Recently, Hernandez and O'Regan [16] started the study of a new class of impulsive equations, by introducing non-instantaneous impulses and proving the solvability of the following system:

$$\begin{cases} x'(t) - Ax(t) = f(t, x(t)), & t \in \bigcup_{i=0}^{N}(s_i, t_{i+1}], \\ x(t) = g_i(t, x(t)), & t \in (t_i, s_i],\ i = 1, \dots, N, \\ x(0) = x_0, \end{cases} \tag{1}$$

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where $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup of linear operators $(T(t))_{t\geq 0}$ defined on a Banach space $(X, \|\cdot\|_X)$, $x_0 \in X$, $0 = t_0 = s_0 < t_1 \leq s_1 < t_2 < \ldots < t_N \leq s_N < t_{N+1} = a$ are fixed numbers, $g_i \in C((t_i, s_i] \times X, X)$ for all $i = 1, \ldots, N$ and $f : [0, a] \times X \to X$ is a suitable function.

As pointed out by the authors, the problem finds its motivation since it models real phenomena in which an impulsive action starts abruptly and stays active on a finite time interval; moreover the authors presented a new technical approach to differential equation problems. Nevertheless, under the set of constraints introduced in [16], the above problem is equivalently described by the following one:

$$\begin{cases} x'(t) - Ax(t) = f(t, x(t)), & t \in \bigcup_{i=0}^{N}(s_i, t_{i+1}] \\ x(t) = \phi(t), & t \in \bigcup_{i=0}^{N}(t_i, s_i] \\ x(0) = x_0, \end{cases} \tag{2}$$

where $\phi : \bigcup_{i=0}^{N}[t_i, s_i] \to X$ is a fixed function (see Proposition 10 in Section 3). System (2) can be also approached by studying the $N + 1$ simultaneous Cauchy problems

$$\begin{cases} x'(t) - Ax(t) = f(t, x(t)), & t \in (s_i, t_{i+1}] \\ x(s_i) = \phi(s_i), \end{cases} \qquad (i = 0, \ldots, N), \tag{3}$$

where $\phi(0) = x_0$. Clearly, this last approach can not be applied for proving the existence of solutions of retarded functional differential equations (RFDEs). This class of equations can be eventually solved by either a global or a sequential approach.

In this article, we prove the existence of bounded solutions of the RFDE problem

$$\begin{cases} x'(t) - A(t)x(t) = f(t, x(t), x(\sigma(t))), & a.e.\ t \in \bigcup_{i=1}^{\infty}(s_i, t_{i+1}] \\ x(t) = (Kx)(t), & t \in [-r, 0] \cup \bigcup_{i=1}^{\infty}(t_i, s_i] \end{cases} \tag{4}$$

with $r > 0$, $t_0 := -r$, $s_0 := 0 < t_1 < s_1 < \ldots < t_i < s_i < \ldots$, and $\lim_{i\to\infty} t_i = +\infty$. Here $A : [0, +\infty) \to \mathcal{L}(X, X)$ is the generator of a uniformly bounded and compact evolution system $E(t, s)$ and $f : [0, +\infty) \times X \times X \to X$ is a function. The operator $K : BC\left(\cup_{i=1}^{\infty}[t_i, s_i]\right) \to BC\left(\cup_{i=1}^{\infty}[t_i, s_i]\right)$ is a bounded and completely continuous operator that represents the initial condition on $[-r, 0]$ and the non-instantaneous impulses on $\bigcup_{i=1}^{\infty}[t_i, s_i]$, $\sigma : [-r, +\infty) \to [0, +\infty)$ is continuous such that $\sigma(t) \leq t$ and describes the delay in the system.

We point out that the time dependence of the operator $A(\cdot)$, the unboundedness of the time interval, the presence of an infinite number of non-instantaneous impulses and of the delay are all novel elements with respect to the previous literature on the topic.

We will seek mild and strong solutions for the system (4) and our method will be based on fixed point and evolution operator theories. We will prove a compactness criterion in order to apply the following well-known fixed point theorem:

Schaefer's Theorem. *Let C be a convex subset of a Banach space E and $0 \in C$. Let $F : C \to C$ be a completely continuous operator, and let*

$$\zeta(F) := \{x \in E : x = \lambda Fx,\ 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

The fact that we will mainly rely on topological conditions represents another difference with respect to the previous papers on the same topic, which are mainly based on metric conditions.