J. Math. Anal. Appl. 441 (2016) 862–884

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Recovering time-dependent inclusion in heat-conductive bodies using a dynamical probe method

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ARTICLE INFO

Article history:
Received 14 August 2015
Available online 13 April 2016
Submitted by H.-M. Yin

Keywords:
Dirichlet-to-Neumann map
Heat probing
Inverse problem

ABSTRACT

We consider an inverse boundary value problem for the heat equation: $\partial_t v = \mathrm{div}_x(\gamma \nabla_x v)$ in $(0,T) \times \Omega$, where Ω is a bounded domain of $\mathbb{R}^3$, and the heat conductivity $\gamma(t,x)$ admits a surface of discontinuity, which depends on time without any spatial smoothness. The reconstruction and, implicitly, the uniqueness of the moving inclusion based on the knowledge of the Dirichlet-to-Neumann operator is achieved using a dynamical probe method according to the construction of fundamental solutions of the elliptic operator $-\Delta + \tau^2 \cdot$, where τ is a large real parameter, and a pair of inequalities relate the data and integrals on the inclusion, in a similar manner to the elliptic case. These solutions depend on the pole of the fundamental solution but also on the large parameter τ, which allows the method to be applied in a very general situation.

1. Introduction

1.1. Inverse heat conductivity problem

Let Ω be a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary $\Gamma = \partial\Omega$, and consider the following initial boundary value problem

$$\begin{cases} \partial_t v = \mathrm{div}_x(\gamma \nabla_x v) & \text{in} \quad \Omega_T = (0,T) \times \Omega, \\ v = f & \text{on} \quad \Gamma_T = (0,T) \times \Gamma, \\ v\big|_{t=0} = v_0 & \text{on} \quad \Omega, \end{cases} \tag{1}$$

where $\gamma = \gamma(t,x) \in L^\infty(\Omega_T) \cap C([0,T]; L^1(\Omega))$, with the following properties.

(C-γ) *A positive function* $(t,x) \mapsto k(t,x)$ *and for all* $t \in [0,T]$, *a non-empty open set* $D(t) \subset \Omega$ exist such that:

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http://dx.doi.org/10.1016/j.jmaa.2016.04.017

- $\frac{1}{C} \leq k \leq C$ in $D_T := \cup_{[0,T]} \{t\} \times D(t)$ for some $C > 1$.
- $k - 1$ has a constant sign in D_T.
- (C-Lip) *The mapping* $t \mapsto D(t)$ *is Lipschitz smooth in the following manner*: $\alpha > 0$ exists such that:

$$\forall t, t' \in [0, T],\ \forall x \in \mathbb{R}^3,\ \mathrm{d}(x, D(t')) \geq \mathrm{d}(x, D(t)) - \alpha |t - t'|.$$

- $\gamma(t, x) = \begin{cases} 1 \quad \text{if} \quad x \notin D(t), \\ k(t, x) \quad \text{if} \quad x \in D(t). \end{cases}$

Remark 1. *We do not assume any smoothness on* $D(t)$ *or that* $\partial D(t) \cap \Gamma = \emptyset$. First, assumption (C-Lip) implies that the Lebesgue measure of the set $D(t') \setminus D(t)$ converges to 0 as $t - t'$ tends to 0 since $D(t') \subset \{x \in \mathbb{R}^3; \mathrm{d}(x, D(t)) \leq \alpha |t - t'|\}$. Second hand, the assumption that γ belongs to $C([0, T]; L^1(\Omega))$ implies that functions such as $t \mapsto \int_\Omega \gamma(t, x) dx$ are continuous. In fact, our assumption on the smoothness of $t \mapsto D(t)$ can be improved, at least for Theorems 3 and 4.

Our main purpose is to study discontinuous perturbations, but we allow $\gamma(t, \cdot)$ to be continuous. Hence, we impose the following assumption.

(C-D) $\inf_{x \in K} |k(t, x) - 1| > 0$ *for any compact set* $K \subset D(t)$, for all $t \in [0, T]$.

We consider a large parameter $\tau > 0$ and allow the initial data $v_0(x)$ to depend on τ under the following condition.

(C-0) τ-independent positive constants C, l_0 exist such that $\|v_0\|_{L^2(\Omega)} \leq C e^{\tau l_0}$, for all τ.

Physically, the region $D(t)$ corresponds to some inclusion in the medium that has different heat conductivity from that in the background domain Ω. The problem addressed in this study is the determination of $D(t)$ using knowledge of the Dirichlet-to-Neumann map (D–N map):

$$\Lambda_{\gamma, v_0} : f \mapsto \partial_\nu v(t, x), \quad (t, x) \in \Gamma_T,$$

where $v = \mathcal{V}(\gamma; f)$ denotes the unique solution of (1), ν is the outer unit normal to Γ, and $\partial_\nu = \dfrac{\partial}{\partial \nu} = \nu \cdot \nabla_x$. In physical terms, $f = f(t, x)$ is the temperature distribution on the boundary and $\Lambda_{\gamma, v_0}(f)$ is the resulting heat flux through the boundary.

The above inverse boundary value problem is related to nondestructive testing where we search for anomalous materials inside a known material.

To clarify our purpose, we briefly recall Ikehata's probe method for the elliptic inverse problem.

1.2. The elliptic situation

In the probe method for the well-known elliptic situation, Problem (1) is replaced by

$$\begin{cases} \mathrm{div}_x (\gamma \nabla_x v) = 0 \quad \text{in} \quad \Omega, \\ v = f \quad \text{on} \quad \Gamma, \end{cases} \tag{2}$$

and D_T is replaced by an open set $D \subset \Omega$. The Dirichlet-to-Neumann operator Λ_γ is a mapping: $H^{\frac{1}{2}}(\Gamma) \ni f \mapsto \partial_\nu v \in H^{-\frac{1}{2}}(\Gamma)$, where v is the unique solution of (2). The probe method (see [7]) starts by considering the fundamental solution $h_0(x) = \frac{1}{4\pi |x - y|}$ of $-\Delta h_0 = \delta_y$, with pole $y \in \Omega$. Then, we approximate h_0 outside a needle $\Sigma \subset \overline{\Omega}$ with one end on Γ and the other is y based on a sequence $\{h_j\}_{j \geq 1}$ such that $-\Delta_x h_j = 0$ in Ω, and we estimate $\int_D |\nabla h_j(x)|^2 dx$ (or $\int_D |\nabla h_0(x)|^2 dx$) according to the following pair of inequalities: