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Asymptotic behaviour of a population model with local, concave growth on Lipschitz domains

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A B S T R A C T

We examine the asymptotic behaviour of $\dot{u} = dAu + f(u)$ for positive initial values $u(0) > 0$. Here A is the generator of an exponentially bounded semigroup on $L^\infty(\Omega)$ with several additional properties. A particular example is the Laplace operator $\Delta_\Omega^{R,\infty}$ with Robin boundary conditions on a Lipschitz domain defined on $L^\infty(\Omega)$. Moreover, $f\colon L^\infty(\Omega) \to L^\infty(\Omega)$ is a local, continuously Fréchet-differentiable, and strictly concave map with $f(0) = 0$. We will analyse the asymptotic behaviour of the solutions as $t \to \infty$ and its dependency on the diffusion coefficient $d > 0$.

1. Introduction

The purpose of the present article is to study the long-time-behaviour of the solution of the following evolution equation:

$$\begin{cases} \dot{u} = dAu + f(u) \\ u(0) = u_0 \in L^\infty(\Omega)_+ \backslash \{0\}. \end{cases}$$

Here $d > 0$ is called the diffusion coefficient, $f\colon L^\infty(\Omega) \to L^\infty(\Omega)$ denotes a continuously Fréchet-differentiable function and A generates a semigroup on $L^\infty(\Omega)$. The detailed assumptions and motivations for this model will be given later (see Assumptions 2.1).

This evolution equation is a generalization of the following population model analysed by Taira in [6] and [7]:

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$$\begin{cases} \dot{u}(t) = d\Delta u(t) + m \cdot u(t) - h \cdot u^2(t) & \text{on } \Omega \text{ for } t \geq 0, \\ \alpha u(t) + \beta \frac{\partial u(t)}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u(0) > 0. \end{cases}$$

Taira assumes that α, β are two non-negative, bounded, and measurable functions with $\alpha + \beta = 1$, m is a bounded and measurable function, h is a non-negative, bounded, and measurable function such that $h > 0$ holds on a set of positive measure. Moreover Ω is assumed to have smooth boundary or is at least of class $C^{2+\theta}$ in the quoted articles. Additional assumptions, which are not of interest for us, can be found in [6] and [7].

This equation is a natural population model with diffusion and logistic growth. Taira was able to give a complete description of the asymptotic behaviour of the solution depending on the diffusion coefficient $d > 0$ in [6] and [7]. More precisely he shows that there exist constants $0 \leq d_1 \leq d_2 \leq \infty$ such that the solution blows up in L^1 in an infinite time if $d \leq d_1$ (i.e. diffusion is too weak to compensate the unbounded growth in some regions), the solution tends to zero if $d \geq d_2$, and the solution converges to a unique positive steady state solution in the intermediate case $d \in (d_1, d_2)$ (i.e. the population is persistent).

The generalization of Taira's model studied here concerns two main points: The underlying domain Ω is allowed to have merely Lipschitz boundary (instead of being smooth) and we replace the non-linearity of logistic growth by a function which is local and concave. Concerning the first generalization, it should be said that it is by no means obvious how to treat Lipschitz domains. In fact, it turns out that the arguments used in [6] and [7] are heavily based on a strong positivity result given in [8, Proposition 3.8], which proof in turn relies on the smoothness of the boundary. In our context we could rephrase this property as follows: The resolvent

$$R(\lambda, \Delta_\Omega^{R,\infty})\colon L^\infty(\Omega) \to D(\Delta_\Omega^{R,\infty})$$

maps positive (i.e. non-negative, non-zero) elements of $L^\infty(\Omega)$ to order units of $D(\Delta_\Omega^{R,\infty})$ for every $\lambda > 0$. Here we denote by $\Delta_\Omega^{R,\infty}$ the Laplace operator with Robin boundary conditions on $L^\infty(\Omega)$ with its usual domain $D(\Delta_\Omega^{R,\infty})$. Being an order unit in $D(\Delta_\Omega^{R,\infty})$ consists essentially of two parts. The behaviour in the interior (i.e. the positivity of the essential infimum in every compact subset of the domain Ω) and the behaviour on the boundary, which depends on the explicit boundary conditions. The proof of the above strong positivity result uses Hopf's maximum principle, which is wrong for Lipschitz domains. To overcome these difficulties we use arguments different from those in [6] and [7].

Our proof is based on a different version of Rabinowitz's global bifurcation theorem given in Theorem C.1 and on Dancer's theorem given in Theorem C.2. The use of the last result is the main difference here and resolves the difficulty concerning the strong positivity result. As a positive side effect the new arguments require less structure of the specific equation. We are now able to treat more general models.

Let us briefly describe the assumptions we make on the generalized model. For details we refer to Assumptions 2.1. We denote by $(T_p)_{p\in[1,\infty)}$ a compatible family of compact, positive, irreducible C_0-semigroups on $L^p(\Omega)$ (for $p \in [1, \infty)$) such that $\|T_p(t)\|_{L^p\to L^\infty} < \infty$ for all $t > 0$. Further, we denote by A_p the generators of these C_0-semigroups and by A the part of A_2 in $L^\infty(\Omega)$, i.e. the operator given by

$$\begin{aligned} D(A) &= \{x \in L^\infty(\Omega)\colon x \in D(A_2) \text{ and } A_2 x \in L^\infty(\Omega)\} \\ Ax &= A_2 x. \end{aligned}$$

This operator generates an exponentially bounded semigroup on $L^\infty(\Omega)$ in the sense of [1, Definition 3.2.5], which will be denoted by T in the following (see Lemma 2.2).