J. Math. Anal. Appl. 439 (2016) 91–102

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Fourier–Bessel heat kernel estimates [☆]

Jacek Małecki, Grzegorz Serafin [*], Tomasz Zorawik

Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, ul. Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

ARTICLE INFO

Article history:
Received 26 September 2015
Available online 24 February 2016
Submitted by S.A. Fulling

Keywords:
Fourier–Bessel heat kernel
Sharp estimate
Bessel process
Transition density

ABSTRACT

We provide sharp two-sided estimates of the Fourier–Bessel heat kernel and we give sharp two-sided estimates of the transition probability density for the Bessel process in $(0, 1)$ killed at 1 and killed or reflected at 0.

1. Introduction

We consider the Fourier–Bessel heat kernel, which is represented in terms of the Bessel functions of the first kind $J_\nu(z)$ and its successive n-th positive zeros $\lambda_{n,\nu}$ in the following way

$$G_t^\nu(x, y) = 2(xy)^{-\nu} \sum_{n=1}^{\infty} \exp\left(-\lambda_{n,\nu}^2 t\right) \frac{J_\nu(\lambda_{n,\nu} x) J_\nu(\lambda_{n,\nu} y)}{|J_{\nu+1}(\lambda_{n,\nu})|^2}, \quad x, y \in (0, 1), \quad t > 0, \tag{1}$$

where $\nu > -1$. The main results of the paper are the following sharp two-sided estimates of $G_t^\nu(x, y)$ given in

Theorem 1.1. *For every $\nu > -1$ we have*

$$G_t^\nu(x, y) \overset{\nu}{\approx} \frac{(1+t)^{\nu+2}}{(t+xy)^{\nu+1/2}} \left(1 \wedge \frac{(1-x)(1-y)}{t}\right) \frac{1}{\sqrt{t}} \exp\left(-\frac{|x-y|^2}{4t} - \lambda_{1,\nu}^2 t\right), \tag{2}$$

whenever $x, y \in (0, 1)$ and $t > 0$.

☆ Jacek Małecki was supported by the National Science Centre grant no. 2013/11/D/ST1/02622. Grzegorz Serafin was financed by the Ministry of Science and Higher Education grant no. IP2012 018472.

* Corresponding author.
E-mail addresses: jacek.malecki@pwr.edu.pl (J. Małecki), grzegorz.serafin@pwr.edu.pl (G. Serafin), tomasz.zorawik@pwr.edu.pl (T. Zorawik).

http://dx.doi.org/10.1016/j.jmaa.2016.02.051
0022-247X/

Here $\overset{\nu}{\approx}$ means that the ratio of the functions on the right- and left-hand sides is bounded from below and above by positive constants depending only on ν. Since the sum in (1) is oscillating, this explicit representation can be only used to examine the behaviour of the kernel for large times. Indeed, it well known that $G_t^\nu(x,y)$ behaves like the first term of the series, whenever $t \geqslant T_0 > 0$ (for every T_0, if we consider the upper bounds and for some T_0, if we deal with the lower bounds). However, the description of the behaviour of $G_t^\nu(x,y)$ for small times is very difficult to obtain from the above-given series representation, since the sum is highly oscillating and the cancellations between the terms matter in that case. This is a reason why we do not use (1) in examining the small-time behaviour, instead we explore the relation between the Fourier–Bessel heat kernel and the transition probability density of the Bessel process with index ν reflected at 0 and killed at 1. This approach enables us to use probabilistic tools like, for example, the Hunt formula or the Strong Markov property, but still the purely analytic studies of the properties of the modified Bessel functions are crucial for the proofs.

The Fourier–Bessel expansions naturally associated with the Fourier–Bessel heat kernel $G_t^\nu(x,y)$ have been studied for a long time in many different contexts, such as the study of the fundamental operators associated with the Fourier–Bessel expansions (see [3–7]) or the related Hardy spaces [8] just to list a few from the latest works (see [12] for more references). Moreover, the Fourier–Bessel expansions are successfully applicable in variety of areas outside Mathematics. The estimates of $G_t^\nu(x,y)$ have been recently studied in [12] and [13], where the provided two-sided estimates of $G_t^\nu(x,y)$ were quantitatively sharp, i.e. the different constants appear in the exponential terms of the lower and upper bounds. It makes the estimates not sharp, whenever $|x-y|^2 >> t$. In the estimates given in Theorem 1.1 the exponential behaviour of the kernel is described explicitly, i.e. the exponential terms in the lower and upper bounds are exactly the same. Such accurate results seem to be quite rare. Notice that even in the classical setting of Laplacian in $\mathbb{R}^n$, the known estimates of related Dirichlet heat kernel for smooth domains (see [17]) are also only quantitatively sharp (see also [16] and the references therein for corresponding results on manifolds). However, in the recent papers [2] and [1] the sharp two-sided estimates for the Dirichlet heat kernel of the half-line (a,∞) associated with the Bessel differential operator have been obtained.

As we have previously mentioned, the result can be equivalently stated in the probabilistic context. More precisely, if we denote by $p_1^{(\nu)}(t,x,y)$ the transition probability density (with respect to the speed measure $m^{(\nu)}(dy) = y^{2\nu+1}dy$) of the Bessel process with index $\nu > -1$ killed at 1 and reflected at 0, then we have $p_1^{(\nu)}(2t,x,y) = G_t^\nu(x,y)$ and consequently

Corollary 1.1. *For given $\nu > -1$ we have*

$$p_1^{(\nu)}(t,x,y) \overset{\nu}{\approx} \frac{(1+t)^{\nu+2}}{(t+xy)^{\nu+1/2}} \left(1 \wedge \frac{(1-x)(1-y)}{t}\right) \frac{1}{\sqrt{t}} \exp\left(-\frac{|x-y|^2}{2t} - \lambda_{1,\nu}^2 t/2\right),$$

whenever $x,y \in (0,1)$ and $t > 0$.

Furthermore, instead of studying the Bessel process reflected at 0, we can impose killing condition at both ends of the interval $(0,1)$. Then we can expend the range of the index of the process to the whole real line ($\mu \in \mathbb{R}$) and denote by $p_{(0,1)}^{(\mu)}(t,x,y)$ the transition probability density (with respect to the speed measure $m^{(\mu)}(dy) = y^{2\mu+1}dy$) of the corresponding process, i.e. the Bessel process killed when it leaves $(0,1)$. Note that for $\mu \geqslant 0$ the process does not hit 0. Consequently the condition at zero (killing or reflecting) is relevant for the considered problem in that case, which means that $p_{(0,1)}^{(\mu)}(t,x,y)$ and $p_1^{(\nu)}(t,x,y)$ are identical for $\mu = \nu \geqslant 0$. Moreover, for $\mu < 0$ we can use the absolute continuity property of the Bessel process with different indices to get that for every $\mu \geqslant 0$ we have

$$p_{(0,1)}^{(-\mu)}(t,x,y) = (xy)^{2\mu} p_{(0,1)}^{(\mu)}(t,x,y), \quad x,y \in (0,1), \quad t > 0.$$

Collecting all together we obtain