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Asymptotic analysis of a semi-linear elliptic system in perforated domains: Well-posedness and correctors for the homogenization limit

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ABSTRACT

In this study, we prove results on the weak solvability and homogenization of a microscopic semi-linear elliptic system posed in perforated media. The model presented here explores the interplay between stationary diffusion and both surface and volume chemical reactions in porous media. Our interest lies in deriving homogenization limits (upscaling) for alike systems and particularly in justifying rigorously the obtained averaged descriptions. Essentially, we prove the well-posedness of the microscopic problem ensuring also the positivity and boundedness of the involved concentrations and then use the structure of the two scale expansions to derive corrector estimates delimitating this way the convergence rate of the asymptotic approximates to the macroscopic limit concentrations. Our techniques include Moser-like iteration techniques, a variational formulation, two-scale asymptotic expansions as well as energy-like estimates.

1. Introduction

We study the semi-linear elliptic boundary-value problem of the form

$$(P^\varepsilon): \begin{cases} \mathcal{A}^\varepsilon u_i^\varepsilon \equiv \nabla \cdot \left(-d_i^\varepsilon \nabla u_i^\varepsilon\right) = R_i\left(u^\varepsilon\right), & \text{in } \Omega^\varepsilon \subset \mathbb{R}^d, \\ d_i^\varepsilon \nabla u_i^\varepsilon \cdot \mathrm{n} = \varepsilon\left(a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon F_i\left(u_i^\varepsilon\right)\right), & \text{across } \Gamma^\varepsilon, \\ u_i^\varepsilon = 0, & \text{across } \Gamma^{ext}, \end{cases} \tag{1.1}$$

for $i \in \{1, \ldots, N\}$ ($N \geq 2$, $d \in \{2, 3\}$). Following [20], this system models the diffusion in a porous medium as well as the aggregation, dissociation and surface deposition of N interacting populations of

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colloidal particles indexed by u_i^ε. As short-hand notation, $u^\varepsilon := (u_1^\varepsilon, \ldots, u_N^\varepsilon)$ points out the vector of these concentrations. Such scenarios arise in drug-delivery mechanisms in human bodies and often include cross- and thermo-diffusion which are triggers of our motivation (compare to [12] for the Sorret and Dufour effects and to [15,29] for related cross-diffusion and chemotaxis-like systems).

The model (1.1) involves a number of parameters: d_i^ε represents molecular diffusion coefficients, R_i represents the volume reaction rate, a_i^ε, b_i^ε are the so-called deposition coefficients, while F_i indicates a surface chemical reaction for the immobile species. We refer to (1.1) as problem (P^ε).

The main purpose of this paper is to obtain corrector estimates that delimitate the error made when homogenizing (averaging, upscaling, coarse graining ...) the problem (P^ε), i.e. we want to estimate the speed of convergence as $\varepsilon \to 0$ of suitable norms of differences in micro–macro concentrations and micro–macro concentration gradients. This way we justify rigorously the upscaled models derived in [20] and prepare the playground to obtain corrector estimates for the thermo-diffusion scenario discussed in [19]. From the corrector estimates perspective, the major mathematical difficulty we meet here is the presence of the nonlinear surface reaction term. To quantify its contribution to the corrector terms we use an energy-like approach very much inspired by [9]. The main result of the paper is Theorem 10 where we state the corrector estimate. It is worth noting that this work goes along the line open by our works [14] (correctors via periodic unfolding) and [25] (correctors by special test functions adapted to the local periodicity of the microstructures). An alternative strategy to derive correctors for our scenario could in principle exclusively rely on periodic unfolding, refolding and defect operators approach if the boundary conditions along the microstructure would be of homogeneous Neumann type; compare with [22] and [28].

The corrector estimates obtained with this framework can be further used to design convergent multiscale finite element methods for the studied PDE system (see e.g. [18] for the basic idea of the MsFEM approach and [7] for an application to perforated media).

It is worth mentioning at this point that the existing literature on corrector estimates justifying the homogenization asymptotics is huge. One of the best studied problems is the derivation of plate theories from bulk elastic bodies with various types of perforations; see for instance the reference monographs [21,26], but also the more recent concrete applications to transport and (static and dynamic) mechanics of membranes as indicated e.g. in [3,4].

This paper is organized as follows: In Section 2 we start off with a set of technical preliminaries focusing especially on the working assumptions on the data and the description of the microstructure of the porous medium. The weak solvability of the microscopic model is established in Section 3. The homogenization method is applied in Section 4 to the problem (P^ε). This is the place where we derive the corrector estimates and establish herewith the convergence rate of the homogenization process. A brief discussion (compare with Section 5) closes the paper.

2. Preliminaries

2.1. Description of the geometry

The geometry of our porous medium is sketched in Fig. 2.1 (left), together with the choice of perforation (referred here to also as "microstructure") cf. Fig. 2.1 (right). We refer the reader to [17] for a concise mathematical representation of the perforated geometry. In the same spirit, take Ω to be a bounded open domain in $\mathbb{R}^d$ with a piecewise smooth boundary $\Gamma = \partial\Omega$. Let Y be the unit representative cell, i.e.

$$Y := \left\{ \sum_{i=1}^{d} \lambda_i \vec{e}_i : 0 < \lambda_i < 1 \right\},$$

where we denote by $\vec{e}_i$ by ith unit vector in $\mathbb{R}^d$.