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A connection between ultraspherical and pseudo-ultraspherical polynomials

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ABSTRACT

The pseudo-ultraspherical polynomial of degree n is defined by $\tilde{C}_n^{(\lambda)}(x) = (-i)^n C_n^{(\lambda)}(ix)$ where $C_n^{(\lambda)}(x)$ is the ultraspherical polynomial. We derive a simple expression linking the polynomials $C_n^{(\lambda)}$ and $\tilde{C}_n^{(\frac{1}{2}-\lambda-n)}$ and show how to derive various properties of the zeros of $\tilde{C}_n^{(\lambda)}$ when $\lambda < 1-n$ from properties of the zeros of $C_n^{(\lambda)}$ when $\lambda > -\frac{1}{2}$.

1. Introduction

The sequence of ultraspherical polynomials belongs to the class of sequences of classical orthogonal polynomials whose orthogonality depends on the values of one or more parameters being constrained to lie within a specified range. The ultraspherical sequence $\{C_n^{(\lambda)}\}_{n=0}^{\infty}$, given by

$$C_n^{(\lambda)}(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m (\lambda)_{n-m}}{m!(n-2m)!}(2x)^{n-2m}, \tag{1.1}$$

is orthogonal on $(-1, 1)$, with respect to the weight function $(1-x^2)^{\lambda-1/2}$, provided $\lambda > -1/2$.

For values of the parameter outside the range that ensures orthogonality, the sequence of ultraspherical polynomials can be defined by the three term recurrence relation that holds when the sequence is orthogonal or by an appropriate generating function, see [2,11]. When $\lambda > -\frac{1}{2}$, $C_n^{(\lambda)}$ has n real, simple zeros which lie in the open interval $(-1, 1)$ and are symmetric about the origin. As λ decreases below $-1/2$, the zeros of $C_n^{(\lambda)}$ depart from the interval $(-1, 1)$ through the endpoints -1 and 1 with an additional two zeros leaving $[-1, 1]$

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each time λ decreases through $-k+1/2$, $k = 1, 2, \dots, \lfloor n/2 \rfloor$. A full description of the departure of the zeros, as well as the kinematics of the collisions of zeros at ± 1, when $\lambda = -3/2, -5/2, \dots$ can be found in [4]. An unusual phenomenon occurs as λ decreases below $-[(n+1)/2]$, namely, the zeros of $C_n^{(\lambda)}$ appear on the imaginary axis with an additional pair of zeros joining the imaginary axis each time λ decreases through successive negative integers. When $\lambda < 1-n$, all n zeros of the polynomial $C_n^{(\lambda)}(x)$ are simple and lie on the imaginary axis (see [4, Th. 3]).

Using the definition of Ismail [9, (20.1.1)] for pseudo-Jacobi polynomials, we define the pseudo-ultraspherical polynomial $\tilde{C}_n^{(\lambda)}$ by

$$\tilde{C}_n^{(\lambda)}(x) := (-i)^n C_n^{(\lambda)}(ix) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(\lambda)_{n-m}}{m!(n-2m)!}(2x)^{n-2m}. \tag{1.2}$$

Since it is known [3, Th. 3] that the polynomial $\tilde{C}_n^{(\lambda)}$ has n real, simple zeros for each fixed $\lambda < 1-n$, a natural question is whether the (finite) sequence $\{\tilde{C}_n^{(\lambda)}\}_{n=1}^{-\lfloor \lambda+1 \rfloor}$ is orthogonal for each fixed $\lambda < 1-n$. The three term recurrence relation satisfied by the sequence of pseudo-ultraspherical polynomials, that follows via a simple substitution from the recurrence relation satisfied by the sequence of ultraspherical polynomials $\{C_n^{(\lambda)}\}_{n=0}^{\infty}$, is given by [5, (4.1)]:

$$(n+1)\tilde{C}_{n+1}^{(\lambda)}(x) = 2x(n+\lambda)\tilde{C}_n^{(\lambda)}(x) + (n+2\lambda-1)\tilde{C}_{n-1}^{(\lambda)}(x). \tag{1.3}$$

One might expect that since $n+2\lambda-1 < 0$ for each $n \in \mathbb{N}$, $n \geq 1$, $\lambda < 1-n$, the sequence $\{\tilde{C}_n^{(\lambda)}\}_{n=1}^{-\lfloor \lambda+1 \rfloor}$ is orthogonal with respect to the weight function $(1+x^2)^{\lambda-1/2}$ for each fixed $\lambda < 1-n$. A positive result in this direction was proved by Richard Askey, [1, (1.8)], in the context of the complex orthogonality of Jacobi polynomials. However, the Askey integral [1, (1.8)] converges for a more restricted range of λ values, namely $\lambda < -n$, and diverges when $-n < \lambda < 1-n$. Although a finite sequence of polynomials that satisfies a three term recurrence relation of appropriate type can be embedded in an infinite sequence of orthogonal polynomials, the restriction $n < 1-\lambda$ that applies in this instance does not make the process meaningful and the measure of orthogonality is not unique since there are infinitely many choices of coefficients in the three term recurrence relation that generates the polynomials of degree greater than n.

The weight function of orthogonality for the ultraspherical sequence $\{C_n^{(\lambda)}\}_{n=0}^{\infty}$ when $\lambda > -\frac{1}{2}$ is $(1-x^2)^{\lambda-1/2}$ while the weight function of orthogonality for the (finite) pseudo-ultraspherical sequence $\{\tilde{C}_n^{(\lambda)}\}_{n=1}^{-\lfloor \lambda+1 \rfloor}$ is $(1+x^2)^{\lambda-1/2}$ provided $\lambda < -n$ so the connection between the two weight functions is to replace x by ix. However, there is no reason to suppose that a connection exists between the zeros of $C_n^{(\lambda)}$ when $\lambda > -\frac{1}{2}$ and the zeros of $\tilde{C}_n^{(\lambda')}$ when $\lambda' < 1-n$.

Note that a proof of the orthogonality of the pseudo-ultraspherical polynomials using differential equations can be found in [5, §2.1]. A proof starting from the Rodrigues formula for ultraspherical polynomials is given in [6].

In Section 2, we derive an identity between the two ultraspherical polynomials $C_n^{(\lambda)}$ and $C_n^{(\frac{1}{2}-\lambda-n)}$. Using this identity, we find a simple and explicit algebraic relationship between the zeros of $C_n^{(\lambda)}$ and those of $\tilde{C}_n^{(\lambda')}$ where $\lambda' = \frac{1}{2} - \lambda - n$. In Sections 3 and 4, we show how our identity can be used to derive monotonicity properties of zeros of pseudo-ultraspherical polynomials from similar properties of the zeros of ultraspherical polynomials (and vice-versa).

2. A connection between $C_n^{(\lambda)}$ and $C_n^{(1/2-\lambda-n)}$

We shall make use of the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ (see, for example [10, Ch. 4] for a definition) in polynomial form with the numerator parameter $a = -n$, $n \in \mathbb{N}$.