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Newton's method for solving generalized equations: Kantorovich's and Smale's approaches

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ABSTRACT

In this paper, we study Newton-type methods for solving generalized equations involving set-valued maps in Banach spaces. Kantorovich-type theorems (both local and global versions) are proved as well as the quadratic convergence of the Newton sequence. We also extend Smale's classical (α, γ)-theory to generalized equations. These results are new and can be considered as an extension of many known ones in the literature for classical nonlinear equations. Our approach is based on tools from variational analysis. The metric regularity concept plays an important role in our analysis.

1. Introduction

It is well-known in the literature of applied mathematics, engineering and sciences that the classical Newton method and its generalizations are among the most famous and effective methods for numerically solving the nonlinear equation $f(x) = 0$, for a given function $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$. This success is due to its quadratic rate of convergence under some suitable assumptions on the problem data and the choice of the initial point. The classical convergence results state that Newton's method is only locally convergent. More precisely, if the function f is sufficiently smooth and its Jacobian $\nabla f(x^*)$ is nonsingular at the solution x^*, then by choosing an initial point x_0 in a neighborhood of this solution x^*, the convergence of the sequence generated by Newton's method is guaranteed and the rate of convergence is at least quadratic. Much more has been written about Newton's (or Simpson–Raphson–Newton's) method and a classical reference is the book by Ortega and Rheinboldt [15].

In 1948, L.V. Kantorovich published a famous paper (see [14,11]) about the extension of Newton's method to functional spaces. This result obtained by L.V. Kantorovich can be regarded both as an existence

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result (of a zero of f) and a convergence result (of the associated iterative process). The assumptions used essentially focus on the values of the function f and its first derivative f' at the starting point x_0 as well as the behavior of the derivative f' in a neighborhood of x_0. Kantorovich's theorem requires the knowledge of a local Lipschitz constant for the derivative. In the same spirit, another fundamental result on Newton's method is the well-known point estimation theory of Smale [19], based on the α-theory and γ-theory for analytical functions. The α-theory uses information about all derivatives of the function f at the initial point x_0 in order to give the size of the attraction's basin around the zero of the function f.

In 1980, S.M. Robinson [17] studied generalized equations with parameters: namely parameterized variational inequalities. He proved an implicit function theorem that can be used to study the local convergence of Newton's method for generalized equations (or variational inclusions). In his Ph.D. thesis, Josephy [10] used this theorem to investigate the local convergence of the Newton and quasi-Newton methods for the variational inequality of the form $f(x) + N_C(x) \ni 0$, where C is a closed convex subset of $\mathbb{R}^m$ and N_C is the forward-normal cone of C. He proved that if the solution x^* is regular (in the sense of Robinson [17]) and if x_0 is in a neighborhood of x^*, then the Josephy–Newton method is well-defined and converges superlinearly to x^*. Further generalization of Newton's method was considered by many authors. For example J.F. Bonnans [3] obtained a local convergence result of Newton's method for variational inequalities under weaker assumptions than the one's required by Robinson's theorem. More precisely, Bonnans proved that under the condition of semi-stability and hemistability (these two conditions are satisfied if Robinson's strong regularity holds at the solution), superlinear convergence of Newton's method (quadratic convergence if f is $C^{1,1}$) holds.

The Josephy–Newton method for set-valued inclusions of the form $f(x)+F(x) \ni 0$, where $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a single-valued mapping of class C^1 and $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is a set-valued map, was also considered by A. Dontchev [5–7]. In this case, the algorithm starts from some point x_0 near a solution and generates a sequence (x_n) defined by solving the following auxiliary problem

$$0 \in f(x_n) + Df(x_n)(x_{n+1} - x_n) + F(x_{n+1}). \tag{1.1}$$

Here (and in what follows), Df indicates the first order derivative of f. In [6,7], Dontchev proved the local Q-quadratic convergence of Josephy–Newton method under the assumptions that f is of class C^1 and F has closed graph with $(f + F)^{-1}$ being Aubin continuous at $(x^*, 0)$ (see [8] for more details).

The present paper is concerned with Newton–Kantorovich and Newton–Smale approaches for generalized equations of the form

$$0 \in f(x) + F(x), \tag{1.2}$$

where $f : X \longrightarrow Y$ is a single-valued mapping defined between the Banach spaces X and Y, supposed to be of class C^2 on an open set U in X, and $F : X \rightrightarrows Y$ is a set-valued mapping with a closed graph.

The paper is organized as follows. In section 2, we recall some preliminary results and backgrounds that will be used in the rest of the paper. Section 3 is devoted to the proof of a Kantorovich-type theorem for the generalized equation (1.2). In section 4, we prove two theorems corresponding to Smale's α and γ-theory for problem (1.2) in the case where f is analytic.

2. Preliminaries

The capital letters $X, Y, \dots$ will denote Banach spaces. By X^* we mean the dual space associated with X. Throughout the paper, we will use the common notation $\|\cdot\|$ for the norm on some arbitrary Banach space X and $\langle \cdot, \cdot \rangle$ for the canonical duality pairing on $X^* \times X$. We denote by $\mathbb{B}_X(x, r)$ (resp. $\mathbb{B}_X[x, r]$) the open (resp. closed) ball with center $x \in X$ and radius $r > 0$. The unit ball in X will be defined by $\mathbb{B}_X$ (or simply