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Asymptotic behaviour and cyclic properties of weighted shifts on directed trees ☆

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ABSTRACT

In this paper we investigate a new class of bounded operators called weighted shifts on directed trees introduced recently in [11]. This class is a natural generalization of the so called weighted bilateral, unilateral and backward shift operators. In the first part of the paper we calculate the asymptotic limit and the isometric asymptote of a contractive weighted shift on a directed tree and that of the adjoint. Then we use the asymptotic behaviour and similarity properties in order to obtain cyclicity results. We also show that a weighted backward shift operator is cyclic if and only if there is at most one zero weight.

1. Introduction

The classes of the so-called weighted bilateral, unilateral or backward shift operators [18,21] are very useful for an operator theorist. Besides normal operators these are the next natural classes on which conjectures could be tested. Recently Z.J. Jabłonski, I.B. Jung and J. Stochel defined a natural generalization of these classes in [11], called weighted shifts on directed trees. Among others, they were interested in hyponormality, co-hyponormality, subnormality etc., and they provided many examples for several unanswered questions. They continued their research in several papers, see [3,4,10,12,23].

In this paper we will study cyclic properties of bounded (mainly contractive) weighted shift operators on directed trees. First, we will explore their asymptotic behaviour, and as an application we will obtain some

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results concerning cyclicity. In the next few pages we give some auxiliary definitions which will be essential throughout this investigation.

1.1. Directed trees

Concerning the definition of a directed tree we refer to the monograph [11]. Throughout this paper $\mathcal{T} = (V, E)$ will always denote a *directed tree*, where V is a non-empty (usually infinite) set and $E \subseteq V \times V \setminus \{(v, v) : v \in V\}$. We call an element of V and E a *vertex* and a *(directed) edge* of $\mathcal{T}$, respectively. If we have an edge $(u, v) \in E$, then v is called a *child* of u, and u is called the *parent* of v. The set of all children of u is denoted by $\mathrm{Chi}_{\mathcal{T}}(u) = \mathrm{Chi}(u)$, and the symbol $\mathrm{par}_{\mathcal{T}}(v) = \mathrm{par}(v)$ stands for u. We will also use the notation $\mathrm{par}^k(v) = \underbrace{\mathrm{par}(\ldots(\mathrm{par}}_{k\text{-times}}(v))\ldots)$ when it makes sense, and par^0 will be the identity map.

If a vertex has no parent, then we call it a *root* of $\mathcal{T}$. A directed tree is either rootless or has a unique root (see [11, Proposition 2.1.1]) which, in this case, will be denoted by $\mathrm{root}_{\mathcal{T}} = \mathrm{root}$. We will use the notation

$$V^\circ = \begin{cases} V \setminus \{\mathrm{root}\} & \text{if } V \text{ has a root,} \\ V & \text{elsewhere.} \end{cases}$$

If a vertex has no children, then we call it a *leaf*, and $\mathcal{T}$ is *leafless* if it has no leaves. The set of all leaves of $\mathcal{T}$ will be denoted by $\mathrm{Lea}(\mathcal{T})$. Given a subset $W \subseteq V$ of vertices, we put $\mathrm{Chi}(W) = \cup_{v\in W} \mathrm{Chi}(v)$, $\mathrm{Chi}^0(W) = W$ and $\mathrm{Chi}^{n+1}(W) = \mathrm{Chi}(\mathrm{Chi}^n(W))$ for all $n \in \mathbb{N}$. The set $\mathrm{Des}_{\mathcal{T}}(W) = \mathrm{Des}(W) = \bigcup_{n=0}^\infty \mathrm{Chi}^n(W)$ is called the *descendants* of the subset W, and if $W = \{u\}$, then we simply write $\mathrm{Des}(u)$. If $n \in \mathbb{N}_0 (:= \mathbb{N} \cup \{0\})$, then the set $\mathrm{Gen}_{n,\mathcal{T}}(u) = \mathrm{Gen}_n(u) = \bigcup_{j=0}^n \mathrm{Chi}^j(\mathrm{par}^j(u))$ is called the *nth generation* of u and $\mathrm{Gen}_{\mathcal{T}}(u) = \mathrm{Gen}(u) = \bigcup_{n=0}^\infty \mathrm{Gen}_n(u)$ is the *(whole) generation* or the *level* of u.

From the equation

$$V = \bigcup_{n=0}^{\infty} \mathrm{Des}(\mathrm{par}^n(u)) \tag{1.1}$$

(see [11, Proposition 2.1.6]), one can easily see that the different levels can be indexed by the integer numbers (or by a subset of the integers) in such a way that if a vertex v is in the kth level, then the children of v are in the $(k + 1)$th level, and whenever $\mathrm{par}(v)$ is defined, it lies in the $(k - 1)$th level.

1.2. Bounded weighted shifts on directed trees

The complex Hilbert space $\ell^2(V)$ is the usual space of all square summable complex functions on V with the standard innerproduct

$$\langle f, g\rangle = \sum_{u\in V} f(u)\overline{g(u)} \qquad (f, g \in \ell^2(V)).$$

For $u \in V$ we define $e_u(v) = \delta_{u,v} \in \ell^2(V)$, where $\delta_{u,v}$ is the Kronecker delta. Obviously the set $\{e_u : u \in V\}$ is an orthonormal base. We will refer to $\ell^2(W)$ as the subspace (i.e. closed linear manifold) $\vee\{e_w : w \in W\}$ for any subset $W \subseteq V$, where the symbol $\vee\{\ldots\}$ stands for the generated subspace.

Let $\boldsymbol{\lambda} = \{\lambda_v : v \in V^\circ\} \subseteq \mathbb{C}$ be a set of *weights* satisfying the following condition: $\sup\Big\{\sqrt{\sum_{v\in\mathrm{Chi}(u)} |\lambda_v|^2} : u \in V\Big\} < \infty$. Then the *weighted shift on the directed tree* $\mathcal{T}$ is the operator defined by

$$S_{\boldsymbol{\lambda}} : \ell^2(V) \to \ell^2(V), \quad e_u \mapsto \sum_{v\in\mathrm{Chi}(u)} \lambda_v e_v.$$