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Formation of boundary layers for singularly perturbed fourth-order ordinary differential equations with the Lidstone boundary conditions

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ABSTRACT

In this paper we concentrate on the aspect of the boundary layer phenomenon that represents a typical asymptotic behavior mode of singularly perturbed systems at the endpoints of the considered interval. We provide a detailed analysis of this phenomenon for the nonlinear fourth-order ordinary differential equations subject to the Lidstone boundary conditions by an analysis of the integral equation associated with an original boundary value problem.

1. Introduction

The present paper is a direct continuation of [6], in which we proved the existence and uniqueness of solutions and constructed their approximation up to exponentially small residuals for the Lidstone boundary value problem (LBVP)

$$\epsilon^2 y^{(4)}(x) + \epsilon(k_1 + k_2)y''(x) + k_1 k_2 y(x) = f(x, y),\ x \in (0, 1) \tag{1}$$

$$y(0) = y(1) = y''(0) = y''(1) = 0, \tag{2}$$

where $k_1 < k_2 < 0$, $0 < \epsilon \ll 1$ are constants, and f is a C^1 function in the area $D_\delta(u)$, specified below in hypothesis (H1). Therefore we frequently use the notations from there and refer to [6] for explanation. We restrict the definitions and results to a bare minimum, necessary to make this paper self-contained and independently readable.

In this paper we focus our attention to a typical phenomenon occurring in singularly perturbed systems, namely the boundary layer phenomenon. In the theory of singular perturbation, a boundary layer is a

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narrow region near the endpoints of the interval under consideration – $x = 0$ and $x = 1$ in our case, where the solution y_ϵ of the LBVP (1), (2) varies rapidly with $y'_\epsilon \to \pm\infty$ at $x = 0$ or/and $x = 1$ for $\epsilon \to 0^+$, and the boundary layer becomes narrower with decreasing ϵ. At the same time y_ϵ converges uniformly to the solution $u = u(x)$ of the reduced problem $k_1k_2y = f(x, y)$, obtained from (1) by the limit process for $\epsilon \to 0^+$, on every compact subinterval of $(0, 1)$. The thickness is typically of order some power of ϵ. We derive the exact formulas describing the asymptotic behavior of the solutions at the endpoints of the interval $[0, 1]$.

The literature on fourth-order differential equations is extensive, due to their many applications in elasticity theory. For an overview of these (and other) applications we refer to the review paper [1] and to references [2,4].

Unfortunately, the sources dealing with the topic of boundary layer phenomena (and a singular perturbation in general) for fourth-order ordinary differential equations are rare; however a detailed analysis of this phenomenon seems to be relevant for the complete understanding of the asymptotic behavior of singularly perturbed systems for $\epsilon \to 0^+$. From these few sources, we can mention Shanthi's and Ramanujam's work [5], motivated by stability problems in fluid dynamics, which presents the asymptotic numerical methods for linear singularly perturbed fourth-order ordinary differential equations of the form

$$-\epsilon y^{(4)} + b(x)y''(x) - c(x)y(x) = -f(x)$$

subject to the boundary conditions

$$y(0) = p,\ y(1) = q,\ y''(0) = -r,\ y''(1) = -s. \tag{3}$$

In a similar way, in [3] the author developed a method for the approximation of the exact solution for the linear problem

$$-\epsilon y^{(4)} - a(x)y''' + b(x)y''(x) - c(x)y(x) = -h(x), \quad (3)$$

near the boundary layers. The above mentioned papers have developed the method for solving a classical problem from computational mathematics in the theory of singular perturbation: namely, to grasp the solution within the boundary layers. To the best of our knowledge, there is no previous work extensively discussing the boundary layer phenomenon for fourth-order ordinary differential equations based on the analytical approach. Most of the papers connected with analysis of the boundary layers are confined to second-order equations, see e.g. [7].

Notation. Throughout our analysis, we will frequently use the notation of the form $h_\epsilon = O(g_\epsilon)$ to indicate that $\lim_{\epsilon\to 0^+} h_\epsilon/g_\epsilon$ is finite, but non-zero.

We start our analysis with an example to illustrate the arising of boundary layers for the LBVP of the form (1), (2), which will be the subject of detailed analysis in the following section.

Example 1. Let us consider the nonlinear LBVP

$$\epsilon^2 y^{(4)} + \epsilon(k_1 + k_2)y'' + \ln(y + e) = x, \quad (2)$$

with $k_1 < k_2 < 0$, $k_1k_2 > 1$. Rewriting the above differential equation to the form (1), that is,

$$\epsilon^2 y^{(4)} + \epsilon(k_1 + k_2)y'' + k_1k_2y = k_1k_2y - \ln(y + e) + x,$$

it is readily verified by calculus that the solution of the reduced problem is $u(x) = e^x - e$. Fig. 1 depicts the formation of boundary layers at $x = 0$. As we can see, the solutions y_ϵ converge uniformly for $\epsilon \to 0^+$