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Diagonal diophantine equations with small prime variables

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ABSTRACT

Let $(a_i, a_j) = 1$, $1 \leq i < j \leq s$ and $s = 2^k + 1$, where $a_1, \cdots, a_s, s$ and $k \geq 4$ are nonzero integers. In this paper, we show that if the diagonal diophantine equation $a_1 p_1^k + \cdots + a_s p_s^k = n$ is satisfying some necessary conditions, then we have the following results: For any $\epsilon > 0$, we have

(i) if $a_1, \cdots, a_s$ are not all of the same sign, then the above equation has solutions in primes p_j satisfying $p_j \ll |n|^{1/k} + A^{3 \cdot 2^{k-1}+\epsilon}$,
(ii) if $a_1, \cdots, a_s$ are all positive, then the above equation is solvable in prime p_j whenever $n \gg A^{3k \cdot 2^{k-1}+1+\epsilon}$.

This result is the general case of the Diophantine equations with Small Prime Variables.

1. Introduction

In 1967, Baker [1] first investigated the problem of small prime solutions of the equations

$$a_1 p_1 + a_2 p_2 + a_3 p_3 = n,$$

satisfying

$$|a_j| p_j \ll |n| + A^C, \tag{1.1}$$

where $(a_i, a_j) = 1$ and a_1, a_2, a_3, n are nonzero integers satisfying some necessary conditions, and $A = \max\{2, |a_1|, |a_2|, |a_3|\}$. This problem was later settled by M.C. Liu and Tsang [16]. Choi [2] proved that we can take $C = 4190$ in (1.1), and M.C. Liu and Wang [19] improved this to $C = 45$, and then Li [12] to $C = 38$. Under the Generalized Riemann Hypothesis, Choi, M.C. Liu, and Tsang [7] obtained $C = 5 + \epsilon$.

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J.Y. Liu and Tsang [18] showed that when the necessary conditions in this problem are replaced by some more restrictive conditions, one can take $C = 17/2$. With the same restrictive conditions as in [18], Choi and Kumchev [4] further reduced this to $C = 20/3$.

In 1991, M.C. Liu and Tsang [17] first studied the quadratic equations

$$a_1 p_1^2 + \cdots + a_5 p_5^2 = n,$$

satisfying

$$p_j \ll |n|^{1/2} + A^C, \tag{1.2}$$

where $(a_i, a_j) = 1$ and $a_1, \cdots, a_5, n$ are nonzero integers satisfying some necessary conditions, and $A = \max\{2, |a_1|, \cdots, |a_5|\}$. Choi and J.Y. Liu [6] first obtained the result for C in (1.2) was $C = 20 + \epsilon$. The number 20 was subsequently reduced to 25/2 by Choi and J.Y. Liu [5] and then to 8 by Choi and Kumchev [3]. The best result so far is due to Harman and Kumchev [8] who showed that $C = 7$.

In 2006, D. Leung [11] studied the cubic equations in the form

$$a_1 p_1^3 + \cdots + a_9 p_9^3 = n,$$

satisfying

$$p_j \ll |n|^{1/3} + A^C, \tag{1.3}$$

where $(a_i, a_j) = 1$ and $a_1, \cdots, a_9, n$ are nonzero integers satisfying some necessary conditions, and $A = \max\{2, |a_1|, \cdots, |a_9|\}$. He obtained $C = 20 + \epsilon$ in (1.3). And Liu [15] reduced this to $C = 14 + \epsilon$.

In this paper, we consider the equations in the case of $k \geq 4$,

$$a_1 p_1^k + \cdots + a_s p_s^k = n, \tag{1.4}$$

where $s = 2^k + 1$ and $a_1, \cdots, a_s, s, k$ are nonzero integers. A necessary condition for the solvability of (1.4) is

$$a_1 + \cdots + a_s \equiv n \pmod 2, \quad (n, a_j) = 1, \quad j = 1, \cdots, s. \tag{1.5}$$

We also suppose

$$(a_i, a_j) = 1, \quad 1 \leq i < j \leq s, \tag{1.6}$$

and write $A = \max\{2, |a_1|, \cdots, |a_s|\}$. The purpose of this paper is to prove the following results. Let ϵ be any positive number.

Theorem 1.1. *Suppose (1.5) and (1.6) hold. If $a_1, \cdots, a_s$ are not all of the same sign, then (1.4) has solutions in primes p_j satisfying*

$$p_j \ll |n|^{1/k} + A^{3 \cdot 2^{k-1} + \epsilon},$$

where the implied constant depends only on ϵ, k and s.