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# Recovering Dirac operator with nonlocal boundary conditions

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ABSTRACT

In this paper inverse problems for Dirac operator with nonlocal conditions are considered. Uniqueness theorems of inverse problems from the Weyl-type function and spectra are provided, which are generalizations of the well-known Weyl function and Borg's inverse problem for the classical Dirac operator.


## 1. Introduction

Problems with nonlocal conditions arise in various fields of mathematical physics [3,6,13,15,24], biology and biotechnology [19,22], and in other fields. Nonlocal conditions come up when values of the functions on the boundary are connected with values inside the domain.

In this paper we study inverse spectral problems for Dirac operator

$$By' + \Omega(x)y = \lambda y, \quad x \in (0, T), \tag{1}$$

endowed with nonlocal linear conditions

$$U_j(y) := \int_0^T y(t)^t d\sigma_j(t) = 0, \quad j = 1, 2. \tag{2}$$

Here

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

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functions $p(x)$ and $q(x)$ are complex-valued and absolutely continuous functions in $(0, T)$, and $\lambda$ is a spectral parameter, vector-valued functions $\sigma_j(t) = \begin{pmatrix} \sigma_{j,1}(t) \\ \sigma_{j,2}(t) \end{pmatrix}$ are complex-valued functions of bounded variations and are continuous from the right for $t > 0$. There exist finite limits $h_{j,i} := \sigma_{j,i}(+0) - \sigma_{j,i}(0),\ i = 1, 2$. Linear forms $U_j$ in (2) can be written as forms

$$U_j(y) := h_{j,1}y_1(0) + h_{j,2}y_2(0) + \int_0^T y(t)^t d\sigma_{j0}(t), \quad j = 1, 2, \tag{3}$$

where vector-valued functions $\sigma_{j0}(t)$ in (3) are complex-valued functions of bounded variations and are continuous from the right for $t \geq 0$, $h_{1,1} \pm ih_{1,2} \neq 0$ and $h_{2,1} \pm ih_{2,2} \neq 0$. We assume that $h_{1,1}h_{2,2} - h_{1,2}h_{2,1} \neq 0$. This condition provides that linear forms $U_j$ $(j = 1, 2)$ are linearly independent.

Classical inverse problems for Eq. (1) with two-point boundary conditions have been studied fairly completely in many works (see [10,11,14,18] and the references therein). Results of the inverse problem for various nonlocal operators can be found in [2,4,5,9,16,17,20,23,25].

In this work, by using Yurko's ideas of the method of spectral mappings [26] we prove uniqueness theorems for the solution of the inverse spectral problems for Eq. (1) with nonlocal conditions (2).

## 2. Main results

Let $X_k(x, \lambda)$ and $Z_k(x, \lambda)$, $k = 1, 2$, be the solutions of Eq. (1) with the initial conditions

$$X_1(0, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = Z_1(T, \lambda),\ X_2(0, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = Z_2(T, \lambda).$$

Denote by $L_0$ the boundary value problem (BVP) for Eq. (1) with the conditions

$$U_1(y) = U_2(y) = 0,$$

and $\omega(\lambda) := \det[U_j(X_k)]_{j,k=1,2}$. The vanishing function $\omega(\lambda)$ is an entire function of exponential type with order 1, and its zeros $\Xi := \{\xi_n\}_{n\in\mathbb{Z}}$ (counting multiplicities) coincide with the eigenvalues of $L_0$. The function $\omega(\lambda)$ is called the characteristic function for $L_0$.

Denote $V_j(y) := y_j(T)$, $j = 1, 2$. Consider the BVP $L_j$, $j = 1, 2$, for Eq. (1) with the conditions $U_j(y) = V_1(y) = 0$. The eigenvalue sets $\Lambda_j := \{\lambda_{nj}\}_{n\in\mathbb{Z}}$ (counting multiplicities) of the BVP $L_j$ coincide with the zeros of the characteristic function $\Delta_j(\lambda) := \det[U_j(X_k), V_1(X_k)]_{k=1,2}$.

For $\lambda \neq \lambda_{n1}$, let $\Phi(x, \lambda) := \begin{pmatrix} \Phi_1(x, \lambda) \\ \Phi_2(x, \lambda) \end{pmatrix}$ be the solution of Eq. (1) under the conditions $U_1(\Phi) = 1$ and $V_1(\Phi) := \Phi_1(T, \lambda) = 0$. Denote Weyl-type function $M(\lambda) := U_2(\Phi)$. It is known [18] that for Dirac operator with classical two-point separated boundary conditions, the specification of the Weyl function uniquely determines the function $\Omega(x)$. However, in the case with nonlocal conditions (2) that describe three-point boundary conditions, it is not true; the specification of the Weyl-type function $M(\lambda)$ does not uniquely determine the function $\Omega(x)$ (see counterexamples in Section 5). For the nonlocal conditions the inverse problem is formulated as follows.

Throughout this paper the functions $\sigma_{ji}(t)$ are known a priori. Assumption condition $S$: $\Lambda_1 \cap \Xi = \emptyset$. The condition $S$ holds in the case $\Omega = 0$ and $\sigma_{j0} = 0$ since the boundary forms $U_j$ $(j = 1, 2)$ are linearly independent. Then the distance between sets of eigenvalues $\Lambda_1$ and $\Xi$ is positive. The condition $S$ holds for small $\Omega$ and $\sigma_{j0}$ since eigenvalues continuously depend on functional parameters of the problem. In general case the set $\Lambda_1 \cap \Xi$ consists of not more than finite number of eigenvalues.