J. Math. Anal. Appl. 440 (2016) 183–219

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Nonlinear generalized sections of vector bundles

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ARTICLE INFO

Article history:
Received 3 August 2015
Available online 17 March 2016
Submitted by R. Gornet

Keywords:
Nonlinear generalized functions
Diffeomorphism invariance
Colombeau algebra
Singular pseudo-Riemannian geometry
Covariant derivative

ABSTRACT

We present an extension of J.F. Colombeau's theory of nonlinear generalized functions to spaces of generalized sections of vector bundles. Our construction builds on classical functional analytic notions, which is the key to having a canonical geometric embedding of vector bundle valued distributions into spaces of generalized sections. This permits to have tensor products, invariance under diffeomorphisms, covariant derivatives and the sheaf property. While retaining as much compatibility to L. Schwartz' theory of distributions as possible, our theory provides the basis for a rigorous and general treatment of singular pseudo-Riemannian geometry in the setting of Colombeau nonlinear generalized functions.

1. Introduction

The theory of distributions, founded by S. Sobolev [60] and L. Schwartz [59], has been very successfully applied in many fields such as the study of partial differential equations, Fourier analysis, engineering and theoretical physics (see, e.g., [1,27,17,66,44]). However, it does not adapt well to nonlinear problems because of the inherent difficulty to define nonlinear operations on distributions.

The problem of *multiplication of distributions* has been approached in various ways which mainly fall into two classes: either one defines the product only for certain pairs of distributions, or one embeds the space $\mathcal{D}'$ of distributions into an algebra. The core issue to overcome in this is a certain algebraic incompatibility between differentiation, multiplication and singular functions that is made precise by L. Schwartz' *impossibility result* [57], which states that there cannot be an associative commutative differential algebra containing $\mathcal{D}'$ as a linear subspace such that the constant function 1 becomes the multiplicative unit and the partial derivatives of distributions as well as the pointwise product of continuous functions are preserved in the algebra.

While this result was commonly so interpreted as to preclude any reasonable multiplication of distributions, one can in fact construct algebras of generalized functions containing $\mathcal{D}'$ by weakening the above requirements in one form or another; see [54,51,24] for a comprehensive overview on what is possible.

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http://dx.doi.org/10.1016/j.jmaa.2016.03.022

A particularly well-known and widely used approach has been introduced by J.F. Colombeau [9,10], who constructed differential algebras of generalized functions containing $\mathcal{D}'$ and preserving the product not of continuous but of *smooth* (i.e., infinitely differentiable) functions.

These *Colombeau algebras*, as they are commonly called, have been developed further [5,11,24,53,51] and applied successfully in a wide variety of fields, ranging from linear and nonlinear partial differential equations with singular data or singular coefficients (see [45] for a recent survey) over pseudodifferential operators and Fourier integral operators with non-smooth symbols [16,28,15,20] to the investigation of topological and algebraic structures in Colombeau generalized function spaces [13,14,4,3,67]. A particular development in the theory of Colombeau algebras concerns a geometric formulation of the theory with the aim of giving a comprehensive framework for problems of non-smooth differential geometry with applications in geophysics, Lie group analysis of differential equations or general relativity [23,25,36,41,35,24,37,64].

In this geometric setting there are several highly interesting, physically relevant results which could not be obtained by distributional methods alone [61,38,62,7,69,42,35,41,63]; however, progress in the geometric theory of Colombeau algebras, in particular the study of generalized sections of vector bundles and generalized pseudo-Riemannian geometry, has been mostly limited to the *special* variant of these generalized function spaces so far [39,40]. This is a simplified variant of the theory which is easier to calculate with, but has several drawbacks which in a sense preclude genuine geometrical results (cf. [24, Section 3.2.2]); in particular, for special Colombeau algebras there is no induced action of diffeomorphisms extending the classical pullback of distributions, there is no canonical embedding of distributions, and no embedding of distributions can commute with arbitrary Lie derivatives.

In contrast to the special variant there also is the so-called *full* variant of Colombeau algebras. There, the drawbacks of the special algebra just mentioned do not appear but are traded in for a more complicated technical apparatus needed for the formulation of the theory. A diffeomorphism invariant local theory, substantially based on previous work of J.F. Colombeau and A. Meril [12] as well as J. Jelínek [32], was for the first time obtained in [23], where the full diffeomorphism invariant algebra $\mathcal{G}^d(\Omega)$ on open subsets $\Omega \subseteq \mathbb{R}^n$ was presented. The generalization to manifolds, which involved a change of formalism because $\mathcal{G}^d(\Omega)$ still was based on the linear structure of $\mathbb{R}^n$, was accomplished in [25] with the introduction of the full Colombeau algebra $\widehat{\mathcal{G}}(M)$ on any manifold M.

In a next step it was naturally very desirable to have an extension to a theory of generalized sections of vector bundles, and in particular a theory of nonlinear generalized tensor fields suitable for applications in (pseudo-)Riemannian geometry. The basic problem encountered in this case is that one cannot use a coordinatewise embedding: simply defining $\widehat{\mathcal{G}}^r_s(M) := \widehat{\mathcal{G}}(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M)$ for generalized (r,s)-tensor fields (where $\mathcal{T}^r_s(M)$ is the space of smooth (r,s)-tensor fields on M) cannot succeed due to a consequence of the Schwartz impossibility result [26, Proposition 4.1]. The underlying reason, which will be detailed in Section 2.1, is that the embedding of distributions into Colombeau algebras always involves some kind of regularization, but in order to regularize non-smooth or distributional sections of a vector bundle one needs to transport vectors between different points of the manifold (cf. [50]); based on these ideas, in [26] an algebra $\bigoplus_{r,s} \widehat{\mathcal{G}}^r_s$ of generalized tensor fields incorporating the necessary modifications was constructed.

This full generalized tensor algebra, however, also suffered from serious drawbacks quite different from those of special algebras:

(i) $\widehat{\mathcal{G}}^r_s$ inherits all the technical difficulties from $\mathcal{G}^d$ and $\widehat{\mathcal{G}}$ and adds even more on top of them, which makes it rather inaccessible for non-specialists and precludes easy applications.

(ii) $\widehat{\mathcal{G}}^r_s$ is not a sheaf: the corresponding proof which worked in all previous algebras (cf., e.g., [23, Section 8]) breaks down due to the failure of the test objects to be ‘localizing’ in a certain sense.

(iii) There is no way to define a meaningful covariant derivative ∇_X on $\widehat{\mathcal{G}}^r_s(M)$ that is $C^\infty(M)$-linear in the vector field X, which would be an indispensable necessity for geometrical applications like the definition of generalized curvature (cf. [49]).