J. Math. Anal. Appl. 440 (2016) 220–239

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The number of limit cycles in perturbations of polynomial systems with multiple circles of critical points ☆

Yanqin Xiong

School of Mathematics and Statistics, Nanjing University of Information Science & Technology, Nanjing, 210044, China

ARTICLE INFO

Article history:
Received 23 October 2015
Available online 10 March 2016
Submitted by J. Shi

Keywords:
Limit cycle
Period annulus
Melnikov function
Hopf bifurcation

ABSTRACT

This paper investigates the problem for limit cycle bifurcations of system $\dot{x} = yF(x,y) + \varepsilon p(x,y)$, $\dot{y} = -xF(x,y) + \varepsilon q(x,y)$, where $F(x,y)$ consists of multiple circles and $p(x,y), q(x,y)$ are polynomials of degree n. The upper bound for the maximal number of limit cycles emerging from the period annulus surrounding the origin is provided in terms of n and the involved multiplicities of circles by using the first order Melnikov function. Furthermore, Hopf bifurcation for a cubic system of this type is discussed.

1. Introduction and main results

Consider a system of the form

$$\dot{x} = yF(x,y) + \varepsilon \sum_{i+j=0}^{n} a_{ij}x^i y^j, \quad \dot{y} = -xF(x,y) + \varepsilon \sum_{i+j=0}^{n} b_{ij}x^i y^j, \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $(a_{ij}, b_{ij}) \in \mathcal{D} \subset \mathbb{R}^{(n+1)(n+2)}$ and $F(x,y)$ is a polynomial in (x,y) with $\mathcal{D}$ bounded and $F(0,0) \neq 0$. Clearly, the set $\{(x,y) \mid F(x,y) = 0\}$ is an invariant set of (1.1)$|_{\varepsilon=0}$, which is formed by singular points. And, system (1.1)$|_{\varepsilon=0}$ has a family of periodic orbits given by

$$L_r: \quad x^2 + y^2 = r^2, \quad r \in (0,\rho), \quad \rho = \min\{\sqrt{x^2+y^2} \mid F(x,y) = 0\},$$

which forms a region called a period annulus denoted by $\mathcal{A}$ surrounding the origin. Then, the Poincaré map or return map corresponding to (1.1) can be expressed as

☆ The work was supported by the Startup Foundation for Introducing Talent of NUIST, No. 2016.
E-mail addresses: yqxiongmath@gmail.com, xiongyanqin13@163.com.

http://dx.doi.org/10.1016/j.jmaa.2016.03.020
0022-247X/

$$\mathcal{P}(r,\varepsilon) = r + \sum_{i\geq 1} M_i(r)\varepsilon^i, \quad r \in (0,\rho), \tag{1.2}$$

where $M_i(r)$, called the ith order Melnikov function, is analytic on $(0,\rho)$. The solution of (1.1) starting from $(r,0)$ is a periodic orbit if and only if $\mathcal{P}(r,\varepsilon) = r$, $r \in (0,\rho)$. By the implicit function theorem, one can investigate the simple zeros of the first non-zero $M_i(r)$ to find the number of limit cycles bifurcating from $\mathcal{A}$, where a limit cycle means an isolated periodic orbit.

From [9,10,1,3,4,6,13,2,11,5,12,7,8], it is well known that

$$M(r^2) := M_1(r) = \oint_{L_r} \frac{\sum\limits_{i+j=0}^{n} b_{ij}x^i y^j dx - \sum\limits_{i+j=0}^{n} a_{ij}x^i y^j dy}{F(x,y)}, \quad r \in (0,\rho). \tag{1.3}$$

By the above discussion, as $M(r^2) \not\equiv 0$, $r \in (0,\rho)$, if $M(r^2)$ has k simple zeros in $r \in (0,\rho)$, then for $\varepsilon > 0$ small enough, the map $\mathcal{P}(r,\varepsilon)$ in (1.2) has at least k fixed points in $(0,\rho)$ such that system (1.1) can have k limit cycles appearing from period annulus $\mathcal{A}$.

From the previous works, there are two main forms on $F(x,y)$ in (1.1) to deal with this problem. One form is $F(x,y) = \prod_{j=1}^{k_1}(x-a_j)\prod_{l=1}^{k_2}(y-b_l)$, where k_1, k_2 are non-negative integers, a_j and b_l are real numbers with $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$. The following cases were studied. One line ($k_1 = 1$, $k_2 = 0$, $a_1 = -1$) in [9]; two parallel lines ($k_1 = 2$, $k_2 = 0$, $a_1 \neq a_2$) in [10]; two orthogonal lines ($k_1 = k_2 = 1$, $a_1 = -a$, $b_1 = -b$) in [1]; three lines, two of them parallel and one perpendicular ($k_1 = 2$, $k_2 = 1$, $a_1 = -a$, $a_2 = -c$, $b_1 = -b$) in [3]; four lines, two of them parallel and the other perpendicular ($k_1 = 2$, $k_2 = 2$, $a_1 = a$, $a_2 = -a$, $b_1 = -b$, $b_2 = b$); the general case in [4]. Another is $F(x,y) = \prod_{j=1}^{m}[(x-a_i)^2 + (y-b_i)^2]^{k_i}$, where $(a_i, b_i) \in \mathbb{R}^2$ and each $k_i \in \mathbb{N}^+$. The case, where $k_i = 1$, $i = 1, 2, \cdots, m$, was considered by [6]; then, the authors [2] extended the result to general k_i. For other particular forms, see [11,5] for one multiple singular line, see [12] for multiple parallel lines, see [7,8] for a conic case and see [13] for $F(x,y) = 1 + x^4$.

To the best of our knowledge, the situation that $\{F(x,y) = 0\}$ consists of multiple circles, whose centers are located at the x-axis, has not been discussed expect for one circle [7,8], and will be considered in this paper. That is, system (1.1) takes the form

$$\begin{aligned} \dot{x} &= y\prod_{i=1}^{m}\left[(x-a_i)^2 + y^2 - b_i^2\right]^{k_i} + \varepsilon \sum_{i+j=0}^{n} a_{ij}x^i y^j, \\ \dot{y} &= -x\prod_{i=1}^{m}\left[(x-a_i)^2 + y^2 - b_i^2\right]^{k_i} + \varepsilon \sum_{i+j=0}^{n} b_{ij}x^i y^j, \end{aligned} \tag{1.4}$$

where k_i, $i = 1, 2, \cdots, m$ are positive integers and each couple $(a_i, b_i) \in \mathbb{R}^2$ with

$$a_i \neq 0, \quad b_i > 0, \quad a_i^2 - b_i^2 \neq 0.$$

If there exists j such that $a_j = 0$, then along the curve L_r, we have

$$(x-a_j)^2 + y^2 - b_j^2 = x^2 + y^2 - b_j^2 = r^2 - b_j^2.$$

Thus, in this case, along the curve L_r, we have

$$\prod_{i=1}^{m}\left[(x-a_i)^2 + y^2 - b_i^2\right]^{k_i} = (r^2 - b_j^2)\prod_{i=1,i\neq j}^{m}\left[(x-a_i)^2 + y^2 - b_i^2\right]^{k_i}.$$

Therefore, we assume that each a_i does not equal zero always.