J. Math. Anal. Appl. 440 (2016) 240–249

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Spectral sets and tiles on vector space over local fields ☆

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ARTICLE INFO

Article history:
Received 27 November 2015
Available online 3 March 2016
Submitted by P.G. Lemarie-Rieusset

Keywords:
Local fields
Tiles
Spectral sets

ABSTRACT

In this article, we prove that a set Ω, which is of positive and finite Haar measure, in K^d is a spectral set with quasi-lattice as a spectrum if and only if it tiles K^d by the dual quasi-lattice as a translation set. Moreover, we obtain some results connected with the dual spectral set conjecture on vector space over local fields.

1. Introduction

A topological field K is called a local field if its topology is locally compact. A non-discrete local field K is connected, then it is either the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$. If K is not connected, then it is totally disconnected. Furthermore, if K is of positive characteristic, then K is a field of formal Laurent series over a finite field $GF(p^c)$. If $c = 1$, it is a p-series field (also known as Vilenkin group in the literature). Note that the 2-series field is also known as the Cantor dyadic group. If $c \neq 1$, then K is an algebraic extension of degree c of a p-series field. If K is of characteristic 0 and is not connected, then K is either the p-adic number field $\mathbb{Q}_p$ or is a finite algebraic extension of such a field. We refer to Theorem 4.12 in [27] for a proof of this classification of local fields.

Let K be a local field with absolute value $|\cdot|$. The ring of integers in K is denoted by $\mathfrak{D}$ and the Haar measure on K is denoted by $\mathfrak{m}$ or dx. We assume that the Haar measure is normalized so that $\mathfrak{m}(\mathfrak{D}) = 1$. The ring $\mathfrak{D}$ is the unique maximal compact subring of K and it is the unit ball $\{x \in K : |x| \leq 1\}$. The ball $\{x \in K : |x| < 1\}$, denoted by $\mathfrak{P}$, is the maximal ideal in $\mathfrak{D}$ and it is principal and prime. The residue class field of K is the field $\mathfrak{D}/\mathfrak{P}$, which will be denoted by $\mathbf{k}$. Let $\mathfrak{p}$ be a fixed element of $\mathfrak{P}$ of maximal absolute value, called a prime element of K. As an ideal in $\mathfrak{D}$, $\mathfrak{P} = (\mathfrak{p}) = \mathfrak{p}\mathfrak{D}$. The residue class field $\mathbf{k}$ is isomorphic to a finite field $\mathbb{F}_q$ where $q = p^c$ is a power of some prime number $p \geq 2$ ($c \geq 1$ being an integer).

☆ This work was supported by the self-determined research funds of Kashgar University (Grant No. (15)2545).
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http://dx.doi.org/10.1016/j.jmaa.2016.02.065
0022-247X/

The dual group $\widehat{K}$ of K is isomorphic to K. We fix a character $\chi \in \widehat{K}$ such that χ is equal to 1 on $\mathfrak{D}$ but is non-constant on $\mathfrak{p}^{-1}\mathfrak{D}$. Then the map $y \mapsto \chi_y$ from K onto $\widehat{K}$ is an isomorphism, where $\chi_y(x) = \chi(yx)$.

For $d \geq 1$, K^d denotes the d-dimensional K-vector space. We endow K^d with the norm

$$|x| := \max_{1\leq j\leq d} |x_j|, \quad \text{for } x = (x_1, \cdots, x_d) \in K^d.$$

The Haar measure on K^d is the product measure $dx_1 \cdots dx_d$ which is also denoted by $\mathfrak{m}$, or $\mathfrak{m}_d$ if it is necessary to point out the dimension. For $x = (x_1, \cdots, x_d) \in K^d$ and $y = (y_1, \cdots, y_d) \in K^d$, we define

$$\langle x, y\rangle := x \cdot y = x_1y_1 + \cdots + x_dy_d.$$

The dual group $\widehat{K}^d$ of K^d consists of all $\chi_y(\cdot)$ with $y \in K^d$, where $\chi_y(x) = \chi(y \cdot x)$.

For local fields and Fourier analysis on them, we can refer to [4,27,28,30].

Let K^d be a d-dimensional K-vector space and $\Omega \subset K^d$ be a Borel set of positive and finite Haar measure. The set Ω is said to be *spectral set* if there exists a set $\Lambda \subset \widehat{K}^d$ such that $\{\chi_\lambda\}_{\lambda\in\Lambda}$ is an orthogonal basis of the space $L^2(\Omega)$ of square Haar-integrable functions. Such a set Λ is called a *spectrum* of Ω and (Ω, Λ) is called a spectral pair.

Let $\Omega \subset K^d$ be a Borel set of positive and finite Haar measure, then a discrete set T is called a *packing set* for Ω, following the terminology in [25], if the intersections $(\Omega + t) \cap (\Omega + t')$ for $t \neq t'$ in T have measure zero. It is a *tiling set* if, in addition, the translates $(\Omega + t)$ cover K^d up to measure zero. We also call (Ω, T) a packing pair and a tiling pair, respectively.

It is clear that, (Ω, T) a tiling pair is equivalent to

$$\sum_{t\in T} 1_\Omega(x - t) = 1$$

for almost all $x \in K^d$, where 1_A denotes the indicator function of a set A.

The Fuglede's conjecture, also called *the spectral set conjecture*, states that Ω is a *spectral set* if and only if Ω tiles K^d. And *the dual spectral set conjecture* states that a discrete set T is a *tiling set* if and only if T is a spectrum.

The spectral set conjecture in $\mathbb{R}^d$ has received considerable attentions during the past decades after publication of the seminal paper of Fuglede [9]. For the case of $\mathbb{R}^d$, many positive results were obtained [12,14,15,17–19] before Tao [29] disproved it by showing that the direction "Spectral ⇒ Tiling" does not hold when $d \geq 5$. Now it is known that the conjecture is false in both directions for $d \geq 3$ [8,20,21,26]. However, it is still open in lower dimensions ($d = 1, 2$). On the other hand, Lagarias and Wang [24] proved that all tilings of $\mathbb{R}$ by a bounded region must be periodic, and that the corresponding translation sets are rational up to affine transformations. This in turn leads to a structure theorem for bounded tiles, which would be crucial for the direction "Tiling ⇒ Spectral". Assume that $\Omega \subset \mathbb{R}$ is a finite union of intervals. The conjecture holds when Ω is a union of two intervals [22]. If Ω is a union of three intervals, it is known that "Tiling ⇒ Spectral" holds, and "Spectral ⇒ Tiling" holds with "an additional hypothesis" [1–3].

Recently spectral measures and their some properties were considered on local fields for the first time in [5]. It was proved recently that the spectral set conjecture is true in the p-adic number fields $\mathbb{Q}_p$ [7,6] and in $\mathbb{Z}_p \times \mathbb{Z}_p$ [13].

In this paper, we consider the spectral set conjecture on d-dimensional K-vector space over local fields K when spectra or tiling sets are quasi-lattices. And then provide some characterizations of measurable sets with finite Haar measure being spectral and tiles.

Theorem 1.1. *Let Ω be a Haar measurable set of K^d with $0 < \mathfrak{m}(\Omega) < \infty$, and let Λ be quasi-lattice in K^d and Λ^* be its dual quasi-lattice. Then (Ω, Λ) is tiling pair if and only if (Ω, Λ^*) is spectral pair.*