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Matrix measure approach to Lyapunov-type inequalities for linear Hamiltonian systems with impulse effect

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ABSTRACT

We present new Lyapunov-type inequalities for Hamiltonian systems, consisting of $2n$-first-order linear impulsive differential equations, by making use of matrix measure approach. The matrix measure estimates of fundamental matrices of linear impulsive systems are crucial in obtaining sharp inequalities. To illustrate usefulness of the inequalities we have derived new disconjugacy criteria for Hamiltonian systems under impulse effect and obtained new lower bound estimates for eigenvalues of impulsive eigenvalue problems.

1. Introduction

It is well-known that the Hamiltonian system of $2n$-linear first-order equations has the form

$$y' = JH(t)y, \quad t \in \mathbb{R},$$

where $y \in \mathbb{R}^{2n}$, H is a $2n \times 2n$ symmetric matrix with piecewise continuous real-valued entries, and J is the so-called symplectic identity given by

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Letting $y = (x, u)^T$ and

$$H(t) = \begin{bmatrix} C(t) & A^T(t) \\ A(t) & B(t) \end{bmatrix},$$

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we may rewrite the Hamiltonian system in an alternative way

$$x' = A(t)x + B(t)u, \quad u' = -C(t)x - A^T(t)u. \tag{1}$$

Although there is an extensive literature on the Lyapunov-type inequalities for second-order differential and difference equations [2,3,8,10,11,13,14], the works on systems are limited and mostly are carried out for planar case [4,17,19]. Second-order equations with impulse effect were first considered in [6], and later the planar Hamiltonian systems under impulse effect have been developed in [5,7] to study disconjugacy and stability problems. To the best of our knowledge, Lyapunov-type inequalities for the general $2n \times 2n$ Hamiltonian system were obtained in [9,16], and nothing is known for the impulsive case.

In the present work we consider (1) under impulse effect, that is,

$$\begin{aligned} &x' = A(t)x + B(t)u, \quad u' = -C(t)x - A^T(t)u, \quad t \neq \tau_i \\ &x(\tau_i^+) = K_i x(\tau_i^-), \quad u(\tau_i^+) = -L_i x(\tau_i^-) + K_i u(\tau_i^-), \\ &t \geq t_0, \quad i \in \mathbb{N} = \{1, 2, \ldots\}, \end{aligned} \tag{2}$$

where

(i) $\{\tau_i\}$ is a strictly increasing sequence of real numbers,
(ii) A, B, and C are $n \times n$ matrices with entries continuous everywhere except possibly at each τ_i, where one-sided limits exist, and $B = B^T$ and $C = C^T$,
(iii) $\{K_i\}$ and $\{L_i\}$ are sequences of $n \times n$ matrices such that each K_i^{-1} for $i \in \mathbb{N}$ exist.

Note that the second-order impulsive equation

$$\begin{aligned} &x'' + C(t)x = 0, \quad t \neq \tau_i \\ &x(\tau_i^+) = K_i x(\tau_i^-), \quad x'(\tau_i^+) = -L_i x(\tau_i^-) + K_i x'(\tau_i^-), \\ &t \geq t_0, \quad i \in \mathbb{N} = \{1, 2, \ldots\} \end{aligned} \tag{3}$$

is equivalent to (2) with $A = 0$ and $B = I$.

By a solution of system (2), we mean $y(t) = (x(t), u(t))$ such that x and u are continuous for all $t \geq t_0$, except possibly at each τ_i, where one-sided limits exist, and that $y(t)$ satisfies the impulsive system (2). A real number c is called a zero (generalized zero) of $x(t)$ if and only if $x(c^-) = \lim_{t \to c^-} x(t) = 0$ or $x(c^+) = \lim_{t \to c^+} x(t) = 0$.

The first result concerning the Hamiltonian system (1) when $n = 1$ was given by Krein [9] for

$$x' = a(t)x + b(t)u, \quad u' = -c(t)x - a(t)u. \tag{4}$$

Theorem 1.1. *Let $b(t) \geq 0$ and $c(t) \geq 0$. If system (4) has a solution $(x(t), u(t))$ with $x(t_1) = x(t_2) = 0$, $x(t) \not\equiv 0$ on (t_1, t_2), then*

$$\int_{t_1}^{t_2} |a(t)| dt + \left(\int_{t_1}^{t_2} b(t) dt \right)^{1/2} \left(\int_{t_1}^{t_2} c(t) dt \right)^{1/2} \geq 2.$$