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Multiple positive radial solutions to some Kirchhoff equations ☆

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ABSTRACT

In this paper, we discuss the existence and multiplicity of positive radial solutions to some Kirchhoff equations and elliptic equations. By using the Leggett–Williams three solutions theorem skillfully, we obtain two positive solutions to Kirchhoff equations under some appropriate conditions. As a direct corollary, we also obtain the same result to elliptic equations. In order to facilitate the proof of main results, we first establish a new simple three solutions theorems for the equation $[a + b\varphi(x)]x = Ax$, where A is a completely continuous operator, $a > 0$, $b \geqslant 0$ and $\varphi(x) \leqslant k(\|x\|)$ for all $x \in P$, where k is a nondecreasing nonnegative continuous function, then apply it to prove the main results. In addition, we discover and prove a new inequality in the article. In the end of the paper, we present an example which makes the elliptic equation has infinite many positive radial solutions and give the approximate images of the nonlinear term.

1. Introduction

In this paper, we study the existence and multiplicity of positive radial solutions to the following Kirchhoff equation

$$\begin{cases} -\left(a + b\displaystyle\int_{B_1} |\nabla u|^2\right)\Delta u = f(u), & x \in B_1, \\ u = 0, & x \in \partial B_1, \end{cases} \tag{1.1}$$

where $B_1 = \{x \in \mathbb{R}^N : |x| < 1\}$, $\partial B_1 = \{x \in \mathbb{R}^N : |x| = 1\}$, the constants $a > 0$, $b \geqslant 0$ and $f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\mathbb{R}_+ := [0, \infty)$.

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Taking $a = 1$ and $b = 0$ in the Kirchhoff equation (1.1), we get the elliptic equation

$$\begin{cases} -\Delta u = f(u), & x \in B_1, \\ u = 0, & x \in \partial B_1. \end{cases} \tag{1.2}$$

In order to make the argument simple, we first establish a abstract three solutions theorem. Let E be a real Banach space with the norm $\|\cdot\|$, P a cone in E [13]. Consider the following operator equation

$$[a + b\varphi(x)]x = Ax, \tag{1.3}$$

where $A : P \to P$ is a completely continuous operator and $a > 0$, $b \geqslant 0$ are two constants. The functional $\varphi : P \to \mathbb{R}_+$ is continuous and satisfies $\varphi(x) \leqslant k(\|x\|)$ for all $x \in P$, where the function $k : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing. By using the Leggett–Williams three solutions theorem, we obtain a new simple three solutions theorem to the equation (1.3) under some appropriate conditions.

In the process of discuss, we first rewrite the Kirchhoff equation (1.1) into an operator equation form like (1.3). Then, by using our new three solutions theorem skillfully, we obtain two positive radial solutions to (1.1) under adding some appropriate conditions to f. Notice that the nonlinearity f in [16] should be increasing in obtaining the multiplicity of solutions to the Kirchhoff equation in one dimension. But here, the nonlinearity f does not need to be increasing thanks to the Leggett–Williams three solutions theorem. It should be emphasized that we also obtain two positive solutions to the elliptic equation (1.2) as a direct corollary, which is relatively difficult to get. In addition, we propose and prove an interesting inequality not seen before. At the end of the article, we give some concrete examples. It is worth mentioning that we give an example which makes the elliptic equation (1.2) has infinite many positive radial solutions and give the approximate images of the nonlinear term.

In the theory and application, the problem of multiple solutions of nonlinear operator equations is very interesting and has already attracted widespread attention of the scholars, see [1,5,6,8,9,15,19]. Among the numerous research results, three solutions theorem is an outstanding multiple solutions theorem, see [2,15,17]. Three solutions theorem has many applications in ordinary differential equations, see [3,7,11,12,18]. But there is less research for the application in partial differential equations, see [4,8,10,14]. What is more, most of the application in partial differential equations was applied on the ring domain, see [4,10], which is similar to the application of ordinary differential equations. We hope that the three solutions theorem could be applied to partial differential equations, especially applied to the multiple radial solutions of partial differential equations. In this case, we can calculate the corresponding Green function and get our desired results by using the properties of Green function. Last year, we discovered that the Leggett–Williams three solutions theorem can be applied to elliptic equations, when taking $b = c$ in the Leggett–Williams three solutions theorem. On this basis, we apply it to Kirchhoff equations. In the process of proofs, we found that there exists a lot of commonality and then abstracted into a new simple three solutions theorem.

The key tool in our approach is the following well known Leggett–Williams three solutions theorem, see [15]. To state this theorem we introduce some notations.

Let E be a real Banach space with the norm $\|\cdot\|$, P be a cone in E. For any $r > 0$, we define $P_r = \{x \in P : \|x\| < r\}$, $\overline{P}_r = \{x \in P : \|x\| \leqslant r\}$ and $\partial P_r = \{x \in P : \|x\| = r\}$. Consider a nonnegative continuous concave functional α defined on P, that is, $\alpha : P \to \mathbb{R}_+$ is continuous and satisfies

$$\alpha((1-t)x + ty) \geqslant (1-t)\alpha(x) + t\alpha(y), \quad x, y \in P, t \in [0, 1].$$

We denote the set $\{x \in P : a \leqslant \alpha(x), \|x\| \leqslant b\}$ by $P(\alpha, a, b)$, where $0 < a < b$. It is clear that $P(\alpha, a, b)$ is a bounded closed convex set. Now we present the Leggett–Williams three solutions theorem.