J. Math. Anal. Appl. 440 (2016) 415–419

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Note

# Navier–Stokes equations with vorticity in Besov spaces of negative regular indices

Zujin Zhang [a,*], Xian Yang [b]

[a] School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, Jiangxi, PR China

[b] Foreign Language Department, Ganzhou Teachers College, Ganzhou 341000, Jiangxi, PR China

ARTICLE INFO

*Article history:*
Received 23 January 2016
Available online 16 March 2016
Submitted by J. Xiao

*Keywords:*
Regularity criteria
Navier–Stokes equations

ABSTRACT

This paper studies the Cauchy problem for the three-dimensional Navier–Stokes equations, and shows that the condition

$$\nabla\times\boldsymbol{u}\in L^{\frac{2}{2-r}}(0,T;\dot{B}^{-r}_{\infty,\infty}),\quad 0<r<2$$

ensures the regularity of the solution on $(0,T)$. This improves and extends many previous results.



## 1. Introduction

We are concerned with the regularity criterion for the weak solutions to the incompressible Navier–Stokes equations (NSE) in $\mathbb{R}^3$:

$$\begin{cases}\partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\triangle\boldsymbol{u}+\nabla\pi=\boldsymbol{0},\\ \nabla\cdot\boldsymbol{u}=0,\\ \boldsymbol{u}(0)=\boldsymbol{u}_0,\end{cases}\tag{1}$$

where $\boldsymbol{u}=(u_1,u_2,u_3)$ and $\pi$ denote the unknown velocity field and pressure of the fluid respectively, and $\boldsymbol{u}_0$ is the prescribed initial data satisfying the compatibility condition $\nabla\cdot\boldsymbol{u}_0=0$. Here and in what follows, we shall use the notations:

$$\partial_t\boldsymbol{u}=\frac{\partial\boldsymbol{u}}{\partial t},\quad \partial_i=\frac{\partial}{\partial x_i},\quad (\boldsymbol{u}\cdot\nabla)=\sum_{i=1}^{3}u_i\partial_i.$$

* Corresponding author.
*E-mail address:* zhangzujin361@163.com (Z. Zhang).

http://dx.doi.org/10.1016/j.jmaa.2016.03.037

The existence of a weak solution of (1) has been established in the pioneer works of Leray [10] and Hopf [6] (for the case of bounded domains). However, the issue of regularity and uniqueness of such a weak solution remains open up to now. The classical Prodi–Serrin conditions (see [3,5,11–14]) say that if

$$\boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=1, \quad 3 \le q \le \infty, \tag{2}$$

then the solution is smooth on $(0,T)$. The difficult limiting case of (2) was recently treated in [3] by using backward uniqueness method.

From the scaling point of view, (2) is important in the sense that

$$\|\boldsymbol{u}_\lambda\|_{L^p(0,t;L^q(\mathbb{R}^3))} = \|\boldsymbol{u}\|_{L^p(0,\lambda T;L^q(\mathbb{R}^3))} \tag{3}$$

for

$$\frac{2}{p}+\frac{3}{q}=1, \quad \text{with} \quad \boldsymbol{u}_\lambda(x,t)=\lambda \boldsymbol{u}(\lambda x, \lambda^2 t)\ (\lambda>0).$$

The Prodi–Serrin conditions (2) were later generalized by Beirãoda Veiga [2] to be

$$\nabla \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2} \le q \le \infty. \tag{4}$$

Notice that (4) with $3/2 \le q \le 3$ was readily followed by (2) and the Sobolev embedding theorems. From the hydrodynamical point of view, (4) is important in the sense that (4) is equivalent to the condition on vorticity for finite $q$,

$$\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2} \le q < \infty. \tag{5}$$

The endpoint case $\boldsymbol{\omega} \in L^1(0,T;L^\infty(\mathbb{R}^3))$ or more general one $\boldsymbol{\omega} \in L^1(0,T;BMO)$ was covered in Kozono and Taniuchi [9]. By further refinement into logarithmic Sobolev inequality related to Besov spaces, Kozono–Ogawa–Taniuchi [7] showed that the following regularity criterion

$$\boldsymbol{\omega} \in L^p(0,T;\dot{B}^0_{q,\infty}), \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2} < q \le \infty. \tag{6}$$

Notice that $L^\infty \subset BMO \subset \dot{B}^0_{\infty,\infty}$, (6) is finest when $q=\infty$.

From the inclusion point of view, we have $L^q \subset \dot{B}^{-\frac{3}{q}}_{\infty,\infty} \cong \dot{F}^{-\frac{3}{q}}_{\infty,\infty}$ $(1 \le q \le \infty)$, and the improvements from classical Lebesgue spaces to Besov spaces (or Triebel–Lizorkin spaces) of negative regular indices should be investigated. By establishing a bilinear estimate of Hölder type in homogeneous Triebel–Lizorkin space, Kozono and Shimada [8] refined (2) as

$$\boldsymbol{u} \in L^{\frac{2}{1-r}}(0,T;\dot{B}^{-r}_{\infty,\infty}), \quad 0<r<1 \tag{7}$$

in a full range; that is,

$$3<q<\infty \Rightarrow 0<r\equiv \frac{3}{q}<1.$$

The corresponding possible extension of (4):