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# An elliptic cross-diffusion system describing two-species models on a bounded domain with different natural conditions ☆

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ABSTRACT

This paper is concerned with an elliptic cross-diffusion system describing two-species models on a bounded domain $\Omega$, where $\Omega$ consists of a finite number of subdomains $\Omega_i$ $(i = 1, \dots, m)$ separated by interfaces $\Gamma_j$ $(j = 1, \dots, m-1)$ and the natural conditions of the subdomains $\Omega_i$ are different. This system is strongly coupled and the coefficients of the equations are allowed to be discontinuous on interfaces $\Gamma_j$. The main goal is to show the existence of nonnegative solutions for the system by Schauder's fixed point theorem. Furthermore, as applications, the existence of positive solutions for some Lotka–Volterra models with cross-diffusion, self-diffusion and discontinuous coefficients are also investigated.



## 1. Introduction

Let $\Omega$ be an open bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with boundary $\partial\Omega$, and let $\Gamma_j$ $(j = 1, \dots, m-1)$ be surfaces, which do not intersect with each other. Assume that $\Omega$ consists of a finite number of subdomains $\Omega_i$ $(i = 1, \dots, m)$ separated by interfaces $\Gamma_j$, and the natural conditions of the subdomains $\Omega_i$ are different. Consider two-species system with cross-diffusion and self-diffusion. Let $u^l = u^l(x,t)$ denote the population density of the $l$th species, $l = 1, 2$. Then by the result of [15], the flux of the population density can be represented as

$$\mathbf{J}^l = -\nabla\left[(k^l(x) + \alpha_1^l(x)u^1 + \alpha_2^l(x)u^2)u^l\right] + d^l u^l \nabla V \quad (x \in \Omega),\ l = 1, 2,$$

where $\nabla$ is the gradient operator, $\alpha_2^1(x)$ and $\alpha_1^2(x)$ are cross-diffusion rates acting between the two species, $V = V(x)$ is the (given) environmental potential describing areas where the environmental conditions are more or less favorable, and for each $l = 1, 2$, $k^l(x)$ and $\alpha_l^l(x)$ are the diffusion rate and the self-diffusion rate of the $l$th species, respectively, and $d^l$ is real constant.

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Since the natural conditions of $\Omega_i$, $i = 1, \dots, m$, are different, then $k^l(x)$, $\alpha_j^l(x)$ and $V(x)$ are allowed to be discontinuous on $\Omega$. Let $\mathbf{q}^l = \mathbf{q}^l(x) := d^l \nabla V$. Assume that

$$\alpha_1^1(x) = \beta^1 k^1(x), \quad \alpha_2^1(x) = \gamma^1 k^1(x), \quad \alpha_1^2(x) = \beta^2 k^2(x), \quad \alpha_2^2(x) = \gamma^2 k^2(x) \quad (x \in \Omega),$$

and that

$$k^l = k^l(x) = k_i^l, \quad \mathbf{q}^l = \mathbf{q}^l(x) = \mathbf{q}_i^l(x) \quad (x \in \Omega_i,\ i = 1, \dots, m),\ l = 1, 2,$$

where for each $l = 1, 2$, $k_i^l$ $(i = 1, \dots, m)$ are positive constants, and $\beta^l$, $\gamma^l$ are nonnegative constants. Thus

$$\mathbf{J}^l = -k_i^l \nabla\big((1 + \beta^l u^1 + \gamma^l u^2)u^l\big) + u^l \mathbf{q}_i^l(x) \quad (x \in \Omega_i,\ i = 1, \dots, m),\ l = 1, 2.$$

By the principle of conservation, the vector function $\mathbf{u} = (u^1, u^2) = (u^1(x,t), u^2(x,t))$ is governed by strongly-coupled reaction–diffusion equations

$$u_t^l = \operatorname{div}\big[k^l(x)\nabla\big((1+\beta^l u^1+\gamma^l u^2)u^l\big) - u^l \mathbf{q}^l(x)\big] + u^l f^l(x, \mathbf{u}) \quad (x \in \Omega_i,\ t > 0,\ i = 1, \dots, m),\ l = 1, 2, \tag{1.1}$$

where $u_t^l = \partial u^l / \partial t$ and

$$f^l(x, \mathbf{u}) = f_i^l(\mathbf{u}) \quad (x \in \Omega_i,\ i = 1, \dots, m).$$

Let $\Gamma := \bigcup_{j=1}^{m-1} \Gamma_j$, and let $\boldsymbol{\nu} = \boldsymbol{\nu}(x) = (\nu_1(x), \dots, \nu_n(x))$ be the unit normal vector to $\Gamma$. Assume that $u^l$ and $\mathbf{J}^l \cdot \boldsymbol{\nu}$ are continuous across the inner boundary $\Gamma$. Then

$$[u^l]_{\Gamma\times(0,+\infty)} = 0, \quad \big[k^l(x)\nabla\big((1 + \beta^l u^1 + \gamma^l u^2)u^l\big) \cdot \boldsymbol{\nu} - u^l \mathbf{q}^l(x) \cdot \boldsymbol{\nu}\big]_{\Gamma\times(0,+\infty)} = 0, \quad l = 1, 2, \tag{1.2}$$

where the symbol $[v]_{\Gamma\times(0,+\infty)}$ denotes the jump in the function $v$ across $\Gamma \times (0, +\infty)$. On the parabolic boundary,

$$u^l = \psi^l(x, t) \quad (x \in \partial\Omega,\ t \ge 0), \quad u^l(x, 0) = u_0^l(x) \quad (x \in \Omega),\ l = 1, 2. \tag{1.3}$$

Problem (1.1)–(1.3) is a parabolic cross-diffusion system. In this paper we consider the corresponding steady-state problem in the form

$$\begin{cases} -\operatorname{div}\big[k^l(x)\nabla\big((1 + \beta^l u^1 + \gamma^l u^2)u^l\big) - u^l \mathbf{q}^l(x)\big] = u^l f^l(x, \mathbf{u}) \quad (x \in \Omega_i,\ i = 1, \dots, m), \\ [u^l]_\Gamma = 0, \quad \big[k^l(x)\nabla\big((1 + \beta^l u^1 + \gamma^l u^2)u^l\big) \cdot \boldsymbol{\nu} - u^l \mathbf{q}^l(x) \cdot \boldsymbol{\nu}\big]_\Gamma = 0, \\ u^l = \psi^l(x) \quad (x \in \partial\Omega), \quad l = 1, 2, \end{cases} \tag{1.4}$$

where $\mathbf{u} = \mathbf{u}(x) = (u^1(x), u^2(x))$.

Based on the pioneering work [15], cross-diffusion systems have been investigated extensively in the literature, and all of the discussions are devoted to the systems with continuous coefficients. The works in [1, 3,5,6,9,10,12–14,20,21] are concerned with elliptic cross-diffusion systems, those in [8,11,19] are for parabolic cross-diffusion systems. On the other hand, the influence of the cross-diffusion is not taken into account in the literature dealing with reaction–diffusion systems with discontinuous coefficients (see [17,18] and the references therein). In this paper problem (1.4) is strongly coupled elliptic systems with cross-diffusion, self-diffusion and discontinuous coefficients. We shall extend the method of Pao [12] for elliptic systems with continuous coefficients to problem (1.4). The aim is to show the existence of nonnegative solutions for problem (1.4) and the existence of positive solutions to some Lotka–Volterra models.