J. Math. Anal. Appl. 437 (2016) 882–901

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Liouville type theorems for stable solutions of the weighted elliptic system

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ARTICLE INFO

Article history:
Received 4 August 2015
Available online 21 January 2016
Submitted by C. Gutierrez

Keywords:
Weighted elliptic system
Stable solutions
Liouville type theorem
Iteration

ABSTRACT

We examine the weighted elliptic system

$$\begin{cases} -\Delta u = (1+|x|^2)^{\frac{\alpha}{2}} v, \\ -\Delta v = (1+|x|^2)^{\frac{\alpha}{2}} u^p, \end{cases} \quad \text{in } \mathbb{R}^N,$$

and prove Liouville type theorems for the classical positive and nonnegative stable solutions in higher dimension. In particular, there are no positive stable solutions for any $N \leq 12 + 5\alpha$, $p > 1$ and $\alpha > 0$. Our proof is based on a combination of the bootstrap argument, Souplet's inequality and intermediate stability criterion, which is used to obtain sharp results.

1. Introduction

We consider the weighted elliptic system

$$\begin{cases} -\Delta u = (1+|x|^2)^{\frac{\alpha}{2}} v, \\ -\Delta v = (1+|x|^2)^{\frac{\alpha}{2}} u^p, \end{cases} \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 5$, $p > 1$, and $\alpha > 0$. We are interested in the Liouville type results—i.e., the nonexistence of the classical positive and nonnegative stable solutions (1.1) in $\mathbb{R}^N$ or the half space $\mathbb{R}^N_+$.

We recall **the case** $\alpha = 0$, the so-called Lane–Emden equations or systems which have been widely studied by many experts. For the second order Lane–Emden equation, the finite Morse index solutions of the nonlinear problem

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http://dx.doi.org/10.1016/j.jmaa.2016.01.032

$$\Delta u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

have been completely classified by Farina (see [7]). Farina also proved that nontrivial finite Morse index solutions to (1.2) exist if and only if $p \geq p_{JL}$ and $N \geq 11$, or $p = \dfrac{N+2}{N-2}$ and $N \geq 3$. Here

$$p_{JL} = \begin{cases} +\infty, & \text{if } N \leq 10, \\ \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}, & \text{if } N \geq 11 \end{cases}$$

is the so-called Joseph–Lundgren exponent (see [10]). His proof made a delicate application of the classical Moser's iteration. There exist many excellent papers to utilize Farina's approach to discuss the second order Hardy–Hénon equation. We refer to [4,20] and the references therein.

In 2013, Liouville theorems for stable solution of the biharmonic equation

$$\Delta^2 u = u^p, \quad \text{in } \Omega \subset \mathbb{R}^N \tag{1.3}$$

were established by the use of the test function and Souplet's inequality in [21,22]. But, they did not obtain the complete classification. To solve the problem, Dávila–Dupaigne–Wang–Wei [5] have derived from a monotonicity formula for solution of (1.3) to reduce the nonexistence of nontrivial entire solutions for the problem (1.3), to that of nontrivial homogeneous solutions, and gave a complete classification of stable solutions and those of finite Morse index solutions. Adopting the similar method, Hu [13] obtained a complete classification of stable solutions and finite Morse index solutions of the fourth order Hénon equation $\Delta^2 u = |x|^\alpha |u|^{p-1}u$.

However, it seems that these above mentioned approaches do not work well with some weighted elliptic systems or negative exponent. There is some new approach dealing with those elliptic equations or systems. *The new approach*, which was obtained by Cowan–Ghoussoub [3] and Dupaigne–Ghergu–Goubet–Warnault [6] independently, is firstly to derive the following interesting intermediate second order stability criterion: for the stable positive solution to (1.3), it holds

$$\sqrt{p} \int_{\mathbb{R}^N} u^{\frac{p-1}{2}} \zeta^2 dx \leq \int_{\mathbb{R}^N} |\nabla \zeta|^2 dx, \quad \forall \zeta \in C_0^1(\mathbb{R}^N).$$

Then this will be carried out through a bootstrap argument which is reminiscent of the classical Moser iteration method. Recently, applying the new approach, the fourth order elliptic equation with positive or negative exponent and the Lane–Emden system have been discussed in [1,11,12,14].

For **the general elliptic system with** $\alpha \neq 0$, Liouville property is less understood and is more delicate to handle than $\alpha = 0$. For example, Fazly [8, Theorem 2.4] utilized an approach which is the test function, Souplet's inequality [19] and the idea of Cowan–Esposito–Ghoussoub in [2], to prove Liouville type result of the system (1.1), provided for any $\alpha \geq 0$ and $p > 1$, $(u, v) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$ is a nonnegative entire stable solution of (1.1) in dimension

$$N < 8 + 3\alpha + \frac{8 + 4\alpha}{p - 1}. \tag{1.4}$$

We recall that the study of radial solutions in [15] suggests the following **conjecture**:

A smooth stable solution to (1.3) exists if and only if $p \geq p_{JL_4}$ and $N \geq 13$.

Moreover, we note from [11,13] that Liouville theorems for the biharmonic equation with negative exponent (the fourth order Hénon equation) hold in space dimension $N \leq 12$ ($N \leq 12 + f(\alpha)$). Consequently,