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The Lagrange and the vanishing discount techniques to controlled diffusions with cost constraints

Héctor Jasso-Fuentes[a,*], Beatris Adriana Escobedo-Trujillo[b], Armando Felipe Mendoza-Pérez[c]

[a] *Departamento de Matemáticas, CINVESTAV–IPN, A. Postal 14-740, México D.F. 07000, Mexico*
[b] *Civil Engineering Faculty, Universidad Veracruzana, Coatzacoalcos, Ver., Mexico*
[c] *Facultad de Ciencias en Física y Matemáticas – UNACH, Boulevard Belisario Domínguez, kilómetro 1081, Sin Número, Terán Tuxtla Gutiérrez, Chiapas, C.P. 29050, Mexico*

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ABSTRACT

In this paper we introduce two useful methods to compute optimal control policies for either the discounted or the average payoff criterion with cost constraints when the dynamic system evolves as a n-dimensional diffusion processes. As for the attribute "cost constraints" we mean the coexistence of a given cost function that in general is dominated above by another function (in particular by a constant) playing the role of an extra constraint in the control model. To deduce optimality results for the discounted case, we employ the Lagrange multipliers technique along with dynamic programming arguments. Then, the vanishing discount method is applied to easily obtain average optimal policies. We support our theory by providing an example on pollution accumulation problem.

1. Introduction

It is well known that the two "basic" infinite-horizon criteria in the most of optimal control problems are the discounted and the average (a.k.a. ergodic) criteria. It is widely recognized that these two criteria have opposite aims; the first one concentrates on the performance of early period of times since the reward (or cost) diminishes in long-periods of time, whereas the second one only focuses on an asymptotic behavior, not taking into account finite intervals. There exist a vast number of manuscripts that are focused on these two criteria providing conditions for the existence of optimal control policies and in some others cases useful implementations on how to compute optimal controls by means of algorithms and pseudocodes. In the most

* Corresponding author. Fax: +52 55 5747 3876.
E-mail addresses: hjasso@math.cinvestav.mx (H. Jasso-Fuentes), bescobedo@uv.mx (B.A. Escobedo-Trujillo), mepa680127@hotmail.com (A.F. Mendoza-Pérez).

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cases, these methods have been applied for the unrestricted case; that is, the unique restrictions on the model are the dynamic system and the set of control policies.

The relation between discounted and average criteria has been mainly studied through the well-known abelian theorems, relating an average functional with its corresponding Laplace transform or resolvent (see, for instance, [17, pp. 7–8] or [36, pp. 180–183]). In particular, in control theory, an interesting question is whether or not a given (optimal) discounted payoff can be used to approximate its (optimal) average counterpart. In many situations the above question is possible, and valuable results on approximations of both optimal values and policies are successfully provided; in particular, when the tool to provide optimality is the dynamic programming technique, the above method is better known as the *vanishing discount method*. Such a method has been widely studied for a variety of controlled systems — see for instance [3,6,9,25,31,33] to quote some of a large number of references.

On the other hand, there exist situations when applications force us to consider additional constraints in the control model. Such constraints often appear as extra costs that are dominated, usually from above, by given functions θ_i, $i = 1, \cdots, l$ (in the most cases θ_i are considered to be constants). Roughly speaking, in this class of control problems, the controller wishes to find a control policy π^*, which belongs to a suitable set Π, in such a way that

$$F(\pi^*) = \sup_{\pi \in \Pi_0} F(\pi), \quad \text{where } \Pi_0 = \{\pi \in \Pi \mid G_i(\pi) \leq \theta_i,\ i = 1, \cdots, l\}, \tag{1.1}$$

for some given functionals F and G_i, $i = 1, \cdots, l$. It is clear that the above problem can also be posed in a minimization mode depending on the context of the application we are dealing with.

Optimal problems with constraints have been extensively studied for several classes of discrete- and continuous-time stochastic systems — see, for instance, [7–10,15,16,18–20,22,21,28,29,32,33,35] among others. All of these references use methods based on dynamic programming, convex programming, and infinite-dimensional linear programming to primarily show the *existence* of optimal controls. It is worth noting, that these previous methods have some advantages and disadvantages when one is dealing with optimal control problems with constraints; for example, the dynamic programming approach provides, among other facts, the basis for the implementations of the so-called policy and value iterations methods, which allow us to finding approximations for both optimal controls and values, however, as far as our knowledge goes, this approach mainly works for the case of *one* cost constraint; that is, when $l = 1$ in the set Π_0 of (1.1) — cf. [15,22,29,32,33]. On the other side, methods such as convex programming and infinite linear programming, allow the use of more cost constraints, but they have the limitation that approximations of either optimal control policies or optimal values may not be successfully obtained since these methods are mainly used to prove only *existence* of optimal results — cf. [7,8,10,18–21].

In this paper we will use the dynamic programming approach but at the same time we want to go a bit further since our goal addresses to the introduction of two useful techniques to easily *compute* optimal values and optimal policies for both criteria (discounted and average payoffs) when a cost constraint is imposed in the model. For this purpose, as a first stage, we first characterize the discounted problem with constraints by means of an equivalent unconstrained problem by using the Lagrange multipliers method so that optimality for the unconstrained yield optimality for the original one when a suitable "multiplier" (to be defined later on) is selected. We provide an easy method to get such multiplier by the only use of elementary differential calculus. Our treatment is very general since it tackles a sufficiently large family of constraints. Next, by knowing optimality of the discounted control problem, we apply the vanishing discount method to the discounted unconstrained problem and hence, with a careful selection of the multipliers, we can deduce optimality results for the average case with cost constraints. As far as our knowledge goes, the only manuscripts addressing separately this approach in continuous time are [29,33].

The remainder of this paper is organized as follows: Section 2 introduces the control system as well as some of our main assumptions. In section 3 we present the discounted control problem with cost constraints and