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Vector valued q-variation for ergodic averages and analytic semigroups

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ABSTRACT

In this paper, we establish UMD lattice-valued variational inequalities for ergodic averages of contractively regular operators and analytic semigroups. These results generalize, on the one hand some scalar-valued variational inequalities in ergodic theory, on the other hand Xu's very recent result on UMD lattice-valued maximal inequality. As an application, we deduce the jump estimates and obtain quantitative information on the rate of the pointwise convergence for semigroups acting on UMD lattice-valued functions.

1. Introduction

Let (Ω,μ) be a measure space and $\mathcal{B}$ be a Banach space. A submarkovian C_0 semigroup $(T_t)_{t\geq 0}$ acting on $L^p(\Omega)$ extends to a semigroup of operators on $L^p(\Omega;\mathcal{B})$, still denoted by $(T_t)_{t\geq 0}$. (In the present paper, any extension of any operator will still be denoted by the same operator whenever no confusion occurs.) Cowling and Leinert showed in [6] that

$$\|T_t f(\omega)-f(\omega)\|\to 0,\ \text{a.e.}\ \omega\in\Omega,\ \text{as}\ t\to 0^+ \tag{1.1}$$

for any separable Banach space $\mathcal{B}$ and any $f\in L^p(\Omega;\mathcal{B})$ with $1<p<\infty$, where $\|\cdot\|$ means taking $\mathcal{B}$-norm. The predecessor of this result is the one [37] by Taggart where the convergence was shown only for Banach spaces having the UMD property, that is, the unconditional martingale difference property (see [32] for the precise definition and related facts).

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When $\mathcal{B}$ is a Banach lattice of measurable functions on a measure space (Σ, ν), $\mathcal{B}$-valued functions on Ω can be viewed as scalar-valued functions on $\Omega \times \Sigma$. We refer the readers to the nice book by Lindenstrauss and Tzafriri [21] for more information on Banach lattices and Banach function spaces. Differently from (1.1), in this paper, we are concerned with the question of whether we have for a.e. $(\omega, \sigma) \in \Omega \times \Sigma$,

$$T_t f(\omega, \sigma) \text{ converges, as } t \to 0^+ \text{ and } t \to \infty \tag{1.2}$$

for any Banach function space $\mathcal{B}$ and any $f \in L^p(\Omega; \mathcal{B})$ with $1 < p < \infty$? *A priori*, the answer seems to be negative, since otherwise Stein's principle [35] implies a vector-valued maximal inequality would hold for all Banach function spaces, which is false since in other words all Banach function spaces would have the H.L. property, contradicting the fact that ℓ^1 does not have the H.L. property (see [26]).

Under the condition that $\mathcal{B}$ is a UMD lattice, Xu [38] has recently established Banach lattice-valued maximal inequalities for ergodic averages of contractively regular semigroups and analytic semigroups, which solved an open question posed in [5]. In this paper, we establish some UMD lattice-valued variational inequalities, which are much stronger than maximal inequalities and immediately imply the pointwise convergence of the underlying family of operators without any more arguments. Moreover, the variational inequalities provide us the quantitative information of the rate of the convergence.

Regarding the largest class of Banach lattices for which vector-valued variational inequality (or pointwise convergence) holds, we have not yet found a characterization. But we have shown in [10] that this class should be strictly smaller than the class satisfying the H.L. property.

The scalar-valued variational inequalities have been the subject of many recent research papers in probability, ergodic theory and harmonic analysis. The first variational inequality was proved by Lépingle [20] for martingales which improves the classical Doob maximal inequality (see also [33] for a different approach and related results). Thirteen years later, Bourgain in [2] proved the variational inequality for the ergodic averages of a dynamic system, which has inaugurated a new research direction in ergodic theory and harmonic analysis. See for instance the references [13,14,3,4,7,16,24–26,28].

Among the works, we would like to mention the papers [22,23,17], where the authors have obtained weighted and ℓ^r $(1 < r < \infty)$ spaces-valued variational estimates for ergodic averages on $\mathbb{R}^n$, as well as the paper [19] where the authors have obtained scalar-valued variational inequalities for general ergodic averages of contractively regular operators and analytic semigroups. The results or ideas in these papers will be exploited in the present paper.

To state our results, let us recall the definition of the q-variation. Given a sequence $(a_n)_{n\geq 0}$ of complex number and a number $1 \leq q < \infty$, the q-variation norm is defined as

$$\|(a_n)_{n\geq 0}\|_{v_q} = \sup\{(|a_{n_0}|^q + \sum_{k\geq 1} |a_{n_k} - a_{n_{k-1}}|^q)^{\frac{1}{q}}\}$$

where the supremum runs over all increasing sequences $(n_k)_{k\geq 0}$ of integers. It is clear that the set v_q of all sequences with finite q-variation norm is a Banach space with respect to the q-variation norm. A continuous analogue of v_q is defined as follows. Given a family $(a_t)_{t>0}$ of complex numbers, define

$$\|(a_t)_{t>0}\|_{V_q} = \sup\{(|a_{t_0}|^q + \sum_{k\geq 1} |a_{t_k} - a_{t_{k-1}}|^q)^{\frac{1}{q}}\}$$

where the supremum runs over all increasing sequences $(t_k)_{k\geq 0}$ of positive real numbers. Then we define V_q to be the Banach space of all $(a_t)_{t>0}$ with finite V_q-norm.

Let (Ω, μ) be a measure space and $\mathcal{B}$ be a Banach lattice of measurable functions on a measure space (Σ, ν). Let $1 < p < \infty$ and $L^p(\Omega; \mathcal{B})$ stands for L^p-space of strongly measurable functions from Ω to $\mathcal{B}$. The functions in $L^p(\Omega; \mathcal{B})$ can be viewed as scalar-valued functions of two variables (ω, σ) on $\Omega \times \Sigma$. For