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Additive maps onto matrix spaces compressing the spectrum

Constantin Costara

Faculty of Mathematics and Informatics, Ovidius University, Mamaia Boul. 124, 900527, Constanţa, Romania

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ABSTRACT

We prove that given a unital C^*-algebra $\mathcal{A}$ and an additive and surjective map $T : \mathcal{A} \to \mathcal{M}_n$ such that the spectrum of $T(x)$ is a subset of the spectrum of x for each $x \in \mathcal{A}$, then T is either an algebra morphism, or an algebra anti-morphism. We arrive at the same conclusion for an arbitrary unital, complex Banach algebra $\mathcal{A}$, by imposing an extra surjectivity condition on the map T.

1. Introduction and statement of results

Let $\mathcal{A}$ be a (complex) unital Banach algebra, and denote its unit by $\mathbf{1}$. By $\sigma(a)$ we shall denote the spectrum of the element $a \in \mathcal{A}$ and $\rho(a)$ will be its spectral radius. A well-known result in the theory of Banach algebras, the Gleason–Kahane–Żelazko theorem, states that if $f : \mathcal{A} \to \mathbf{C}$ is $\mathbf{C}$-linear (that is, additive and homogeneous with respect to complex scalars) and $f(a) \in \sigma(a)$ for every $a \in \mathcal{A}$, then f is multiplicative. (See e.g. [5] and [6].) Kowalski and Slodkowski generalized their result in [7], by proving that if $f : \mathcal{A} \to \mathbf{C}$ with $f(0) = 0$ satisfies

$$f(x) - f(y) \in \sigma(x - y) \qquad (x, y \in \mathcal{A}), \tag{1}$$

then f is automatically $\mathbf{C}$-linear, and therefore also multiplicative. (That f is $\mathbf{R}$-linear and the fact that $f(ia) = if(a)$ for all $a \in \mathcal{A}$ come automatically from the inclusions (1), which combine spectrum-preserving properties and additivity properties on the functional f.) In particular, if $f : \mathcal{A} \to \mathbf{C}$ is additive and $f(x) \in \sigma(x)$ for every $x \in \mathcal{A}$, then f is a character of $\mathcal{A}$.

The natural extension of the Gleason–Kahane–Żelazko theorem for the case when the range $\mathbf{C}$ of f is replaced by $\mathcal{M}_n$, the algebra of $n \times n$ matrices over $\mathbf{C}$, was obtained by Aupetit in [1].

E-mail address: cdcostara@univ-ovidius.ro.

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Theorem 1. *(See [1, Theorem 1].) If $T : \mathcal{A} \to \mathcal{M}_n$ is a surjective* **C***-linear map such that*

$$\sigma(T(x)) \subseteq \sigma(x) \qquad (x \in \mathcal{A}), \tag{2}$$

then either

$$T(xy) = T(x)T(y) \qquad (x, y \in \mathcal{A}) \;\; or \;\; T(xy) = T(y)T(x) \qquad (x, y \in \mathcal{A}). \tag{3}$$

In fact, [1, Theorem 1] states that if $T : \mathcal{A} \to \mathcal{M}_n$ is linear, unital and onto, sending invertible elements from $\mathcal{A}$ into invertible elements of $\mathcal{M}_n$, then T is of the form (3). If (2) holds, then $x \in \mathcal{A}$ invertible implies $0 \notin \sigma(x)$, thus by (2) we have $0 \notin \sigma(T(x))$, which means that the matrix $T(x)$ is invertible. By Lemma 6 we also have that T sends the unit element of $\mathcal{A}$ into the unit element of $\mathcal{M}_n$. (See also [4, Theorem 2.2].) Thus, under the hypothesis of Theorem 1 we have that T is unital and invertibility-preserving.

Under the hypothesis of Theorem 1, the map T is either an algebra morphism, or an algebra anti-morphism. In this paper, we study the same type of problem as the one considered by Theorem 1, assuming only additivity instead of linearity over the complex field **C**. Our first result states that if $\mathcal{A}$ is supposed to be a C^*-algebra, then we arrive at the same conclusion by assuming only additivity on T.

Theorem 2. *Let $\mathcal{A}$ be a unital C^*-algebra and suppose $T : \mathcal{A} \to \mathcal{M}_n$ is a surjective additive map such that (2) holds. Then T is of the form (3).*

As a corollary, we obtain the following generalization of [1, Theorem 2] for the case of additive maps defined on C^*-algebras which compress the spectrum.

Theorem 3. *Let $\mathcal{A}$ be a unital C^*-algebra, and let $\mathcal{B}$ be a complex, unital Banach algebra having a separating family of irreducible finite-dimensional representations. Suppose $T : \mathcal{A} \to \mathcal{B}$ is additive and onto such that (2) holds. Then T is a Jordan morphism, that is*

$$T(x^2) = T(x)^2 \qquad (x \in \mathcal{A}).$$

For the general case of an arbitrary Banach algebra $\mathcal{A}$, we shall impose an extra surjectivity assumption on the map T in order to obtain the same type of result.

Theorem 4. *Let $\mathcal{A}$ be a unital Banach algebra and suppose $T : \mathcal{A} \to \mathcal{M}_n$ is a surjective additive map such that (2) holds. Suppose also that there exist $x_1, \ldots, x_{n^2} \in \mathcal{A}$ such that*

$$\{T(x_1) + T(ix_1)/i, \ldots, T(x_{n^2}) + T(ix_{n^2})/i\} \subseteq \mathcal{M}_n \tag{4}$$

are linearly independent over **C**. *Then T is of the form (3).*

We do not know whether the assumption that the matrices in (4) span $\mathcal{M}_n$ over the complex field may be removed from the statement of Theorem 4. We believe that this hypothesis can be eliminated, being a consequence of the fact that T is surjective and that (2) holds, but we have not been able to prove it. An important part of the proof of Theorem 4 can be carried out without the surjectivity hypothesis given by (4) being assumed, using only the surjectivity of the map T. See also the final remark in Section 3.

2. Preliminaries

Throughout this section, $\mathcal{A}$ will denote an arbitrary unital Banach algebra. The first result shows that, as in the **C**-linear case [2, Theorem 5.5.2], under the hypothesis of Theorem 2 we have that the continuity of the map T is automatic.