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Instability of standing waves for inhomogeneous Hartree equations

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ABSTRACT

We are concerned with the instability of standing-wave solutions $e^{i\omega t}\varphi_\omega(x)$ to the Hartree equation $iu_t + \Delta u + gu(|\cdot|^{-\gamma} * (g|u|^2)) = 0$ for some suitable $g(x)$, where φ_ω is a ground state solution. The instability of standing-wave solutions reveals a balance between ω and γ.

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1. Introduction

In this paper, we consider the following Hartree equation

$$\begin{cases} iu_t + \Delta u + gu(|x|^{-\gamma} * (g|u|^2)) = 0, \quad u : \mathbb{R}^N \times [0,T) \to \mathbb{C} \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $*$ denotes the convolution in $\mathbb{R}^N$ and we assume that $N \geq 3$. We consider the following assumptions for the function g:

$$(g_1): \quad g(x) \geq 0, \quad g(x) \not\equiv 0, \quad g \in C^2(\mathbb{R}^N \setminus \{0\}, \mathbb{R}), \quad g \in L^\theta_{loc}(\mathbb{R}^N), \ \theta > \frac{2N}{4-\gamma};$$

$$(g_2): \quad \exists C > 0, 2b < 4-\gamma < 2a \ \text{ s.t. } \left| x^\alpha \partial_x^\alpha (g(x) - \frac{1}{|x|^b}) \right| \leq \frac{C}{|x|^a} \ \text{ for } \ |x| \geq 1, |\alpha| \leq 2,$$

where $0 < b < 2$ and $0 < \gamma < \min\{N, 4-2b\}$. Note that $g(x) = |x|^{-b}$ satisfies (g_1) and (g_2).

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The well-known Choquard equation $iu_t+\Delta u+(|\cdot|^{-1}*|u|^2)u=0$ can be seen as a special case of the generalized nonlocal Schrödinger equation (1.1) with $g(x)\equiv 1$ and $\gamma=1$. Equation (1.1) arises in various domains of mathematical physics, such as the physics of multiple-particle systems, quantum mechanics, physics of laser beams [4,8,19,20], etc. The Choquard equation is a time-dependent version of some equations proposed by Choquard and has been investigated extensively in the context of $H^1(\mathbb{R}^3)$ [2,16]. We remark that the nonlinear Choquard equation is a certain approximation to Hartree–Fock theory for a one component plasma [11].

By a standing-wave solution of (1.1), we mean a solution of the form $u(x,t)=e^{i\omega t}\varphi_\omega(x)$ ($\omega>0$ being a constant), where $\varphi_\omega\in H^1(\mathbb{R}^N)$ satisfies the stationary equation

$$-\Delta\varphi_\omega+\omega\varphi_\omega-g(x)(|\cdot|^{-\gamma}*(g\varphi_\omega^2))\varphi_\omega=0. \tag{1.2}$$

The existence of positive solutions of (1.2) with $g(x)\equiv 1$ and $N=3$ has been studied by Lieb [10,12] and Lions [13,14,17]. For general dimensions, Ma and Zhao [16] show that every positive solution to (1.2) is spherically symmetric and monotone decreasing about some point. Moreover, it has been shown in [18] that the ground state solutions have asymptotic decay rate $u(x)\sim O(e^{-\mu|x|})$ for some $\mu>0$. In [10], Lieb established the uniqueness of the ground state solution with $\gamma=1$, $g\equiv 1$ and $N=3$. Then in [9] the author proved that the uniqueness of the ground state can be adapted to $N=4$, while for the general Hartree equation (1.1), the uniqueness of the Ground-state solution is still an open problem.

To study the Hartree equations, an important issue related to whether or not the global existence can be obtained for arbitrary classes of initial data is the stability of the standing-wave solutions. A large amount of work has been devoted to this subject (see for example, [1,3–7,15,21,22] and the references therein). Roughly speaking, the orbital stability means that if the initial data is close to the set of standing-wave solutions, then the solution exists for all $t\geq 0$ and stays close to the same set for all t. Both necessary and sufficient conditions for stability in this sense have been established in [6,7] for a general class of Hamiltonian system.

We note that, the Hartree equation (1.1) possesses a nonlocal nonlinearity bringing singularity near $x=y$, which differs from the classical power-type local case

$$\begin{cases} iu_t+\Delta u+|u|^{p-1}u=0, \quad (x,t)\in\mathbb{R}^N\times\mathbb{R}, \\ u(x,0)=u_0(x)\in H^1(\mathbb{R}^N). \end{cases} \tag{1.3}$$

Cazenave [1] established a variational framework to study the stability and instability of the standing waves for (1.3), and proved that the ground state solution of (1.3) is stable for all $\omega>0$ if $p<1+4/N$, while is unstable for all $\omega>0$ if $1+4/N<p<1+4/(N-2)$. The instability of the bound state solution with $p=1+4/N$ was proved by Weinstein [22].

For inhomogeneous Schrödinger equations $iu_t+\Delta u-k(x)|u|^{p-1}u=0$, Fibich, Liu and Wang [3,15] have proved the stability and instability of standing-wave solutions for $k(\epsilon|x|)$ with ϵ small and $k\in C^4\cap L^\infty$. Based on Hardy inequalities, Fukuizumi and Ohta [5] obtained the instability of the standing-wave solutions for a small $\omega>0$ with the inhomogeneity k of nonlinearity behaving like $|x|^{-b}$ at infinity with $0<b<2$. It was shown that if $1+(4-2b)/N<p<1+(4-2b)/N-2$, $N\geq 3$, then the standing-wave solutions are orbitally unstable for sufficiently small $\omega>0$. The method they used differs from the classical method, since the latter one depends deeply on the convexity of the mass functional, which may not exist in view of the existence of an inhomogeneous form.

For Hartree equations of the homogeneous case ($g\equiv 1$) of (1.1), similar results can be obtained by Cazenave's framework and Strauss's methods. More precisely, for all $\omega>0$, we are able to show the stability results with $0<\gamma<2$, and the instability results with $2\leq\gamma<4$. Once again, their methods will fail for the inhomogeneous case when $g(x)$ is not identical to 1. In this paper, we adapt a sufficient condition for the instability as was used in [5]. In fact, a instability lemma will avoid us to construct a blow-up sequence only to show a weak instability.

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