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Pointwise eventually non-expansive action of semi-topological semigroups and fixed points

Massoud Amini [a], Alireza Medghalchi [b], Fouad Naderi [a,*]

[a] Department of Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Tehran 14115-134, Iran
[b] Faculty of Mathematical Sciences and Computer, Kharazmi University, 50 Taleghani Avenue, Tehran, Iran

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ABSTRACT

In this paper we define pointwise eventually non-expansive action of semi-topological semigroups and prove fixed point theorems for it. As a result we give an affirmative answer (without any extra condition on the Banach space) to an open problem raised by Kirk and Xu (2008) [7]. Our results also extend and improve fixed point theorems of Lim and Lau–Mah.

1. Introduction

Let K be a subset of a Banach space E. A self-mapping T on K is said to be *non-expansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in K$. If H and K are non-empty subsets of a Banach space E and H is bounded, for $k \in K$, define $r(H, k) = \sup\{\|h - k\| : h \in H\}$. Put $r(H, K) = \inf\{r(H, k) : k \in K\}$ and let $C(H, K) = \{k \in K : r(H, k) = r(H, K)\}$. When K is convex, we say that K has *normal structure* if for each bounded closed convex subset W of K with more than one point, there exists an $x \in W$ such that $r(W, x) < \delta(W) = \mathrm{diam}(W)$, or equivalently, $C(W, W)$ is a proper subset of W. In 1965 a fundamental existence theorem for non-expansive mappings was proved by Kirk [6]:

Theorem 1.1. *Let K be a non-empty weakly compact convex subset of a Banach space E with normal structure, and let T be a non-expansive self-mapping on K. Then T has a fixed point in K.*

* Corresponding author.
E-mail addresses: mamini@modares.ac.ir (M. Amini), a_medghalchi@khu.ac.ir (A. Medghalchi), f.naderi@modares.ac.ir (F. Naderi).

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Alspach [1] has shown that the condition of normal structure in the above theorem is necessary. Kirk's theorem can be further generalized for some semigroups of non-expansive self-mappings on K by the following considerations.

Let S be a *semi-topological semigroup*, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \mapsto sa$ and $s \mapsto as$ from S into S are continuous. S is called *left (right) reversible* if any two closed right (left) ideals of S have non-void intersection.

An action of S on a subset K of a topological space E is a mapping $(s, x) \mapsto s(x)$ from $S \times K$ into K such that $(st)(x) = s(t(x))$ for $s, t \in S$, $x \in K$. The action is separately continuous if it is continuous in each variable when the other is kept fixed. Every action of S on K induces a representation of S as a semigroup of self-mappings on K denoted by $\mathcal{S}$, and the two semigroups are usually identified. When the action is separately continuous, each member of $\mathcal{S}$ is continuous. We say that S has a common fixed point in K if there exists a point x in K such that $sx = x$ for all $s \in S$. When E is a normed space, the action of S on K is *non-expansive* if $\|s(x) - s(y)\| \leq \|x - y\|$ for all $s \in S$ and $x, y \in K$.

The following result of Lim [10] is a generalization of Kirk's theorem for semi-topological semigroups:

Theorem 1.2. *Let K be a non-empty, weakly compact, convex subset of a Banach space E with normal structure and let S be a left reversible semi-topological semigroup of non-expansive, separately continuous action on K. Then S has a common fixed point in K.*

There are two different actions for a semi-topological semigroup which are of interest, i.e., asymptotically and semi-asymptotically non-expansive actions (see [2] for more details). The action of a semi-topological semigroup S on a subset K of a Banach space E is *right (left) asymptotically non-expansive* if for each $x, y \in K$ there is a left (right) ideal $J \subseteq S$ such that $\|s(x) - s(y)\| \leq \|x - y\|$ for $s \in J$ [5]. A special case of semi-asymptotically non-expansive action was used by Kirk and Xu in [7]:

Definition 1.3. A mapping $T : K \to K$ is said to be pointwise eventually non-expansive if for each $x \in K$ there exists $N(x) \in \mathbb{N}$ such that if $n \geq N(x)$,

$$\|T^n(x) - T^n(y)\| \leq \|x - y\| \text{ for all } y \in K.$$

Kirk and Xu then posed the following question [7]:

Question. Does a pointwise eventually non-expansive mapping defined on a weakly compact convex subset K of a Banach space X have a fixed point if X satisfies either of the following conditions:

1- X is nearly uniformly convex?
2- the asymptotic center relative to K of each sequence in K is compact?

In section 2 we define pointwise eventually non-expansive action for semi-topological semigroups which extends the Definition 1.3 of Kirk and Xu. Notice that in [2] a different term, i.e., semi-asymptotically non-expansive action has been used for pointwise eventually non-expansive action but in fact they are the same. In section 3 we show that there is a fixed point for pointwise eventually non-expansive action of a left reversible semi-topological semigroup S when S acts on a weakly compact convex subset of a Banach space with normal structure. As a result we give a positive answer to the open problem posed by Kirk and Xu without any extra conditions on the Banach space in question, at the same time our theorem improves Lim's fixed point Theorem 1.2. Section 4 deals with a generalization of our fixed point theorem for weak* topology and extends the Lau–Mah fixed point theorem. In this section the underlying space is the Fourier algebra of a compact separable group and it turns out that condition 2 of Kirk–Xu question must be satisfied somehow in this case. Our theorems are new and are not results of any previous work.