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Considering copositivity locally

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ABSTRACT

We say that a symmetric matrix A is copositive if $\mathbf{v}^T A\mathbf{v} \geq 0$ for all nonnegative vectors $\mathbf{v}$. The main result of this paper is a characterization of the cone of feasible directions at a copositive matrix A, i.e., the convex cone of symmetric matrices B such that there exists $\delta > 0$ satisfying $A+\delta B$ being copositive. This cone is described by a set of linear inequalities on the elements of B constructed from the so called set of (minimal) zeros of A. This characterization is used to furnish descriptions of the minimal (exposed) face of the copositive cone containing A in a similar manner. In particular, we can check whether A lies on an extreme ray of the copositive cone by examining the solution set of a system of linear equations. In addition, we deduce a simple necessary and sufficient condition for the irreducibility of A with respect to a copositive matrix C.

1. Introduction

Let $\mathcal{S}^n$ be the vector space of real symmetric $n \times n$ matrices. A matrix $A \in \mathcal{S}^n$ is called *copositive* if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n_+$, where $\mathbb{R}^n_+$ denotes the set of element-wise nonnegative n-vectors. The set of copositive matrices forms a convex cone, the *copositive cone*, $\mathcal{COP}^n$. This matrix cone is of interest for combinatorial optimization, for surveys see [7,10,12,17]. It is a classical result by Diananda [8, Theorem 2] that for $n \leq 4$ the copositive cone can be described as the sum of the cone of positive semi-definite matrices $\mathcal{S}^n_+$ and the cone of element-wise nonnegative symmetric matrices $\mathcal{N}^n$. In general, this sum is a subset of the copositive cone, $\mathcal{S}^n_+ + \mathcal{N}^n \subset \mathcal{COP}^n$ (we use $\mathcal{A} \subset \mathcal{B}$ to denote that $\mathcal{A}$ is a subset of $\mathcal{B}$, but not necessarily a strict subset). Prof. Alfred Horn showed that for $n \geq 5$ the inclusion is strict [8, p. 25].

In this contribution we investigate properties of the copositive cone related to convex analysis. In particular, we consider the minimal faces of copositive matrices and irreducibility of copositive matrices with respect to other copositive matrices, and relate these to the set of their zeros or their minimal zeros.

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A vector $\mathbf{u} \in \mathbb{R}^n_+$ whose elements sum up to one is called a *zero* of a copositive matrix A if $\mathbf{u}^T A\mathbf{u} = 0$. Note that for all $\lambda > 0$ we have $\mathbf{u}^T A\mathbf{u} = 0$ if and only if $(\lambda\mathbf{u})^T A(\lambda\mathbf{u}) = 0$, and thus the requirement that the elements of $\mathbf{u}$ sum up to one is simply a normalization. In the literature, sometimes other normalizations are used or no normalization is used or instead of a normalization the authors require that $\mathbf{u} \neq \mathbf{0}$. However, it is a relatively trivial matter to transfer between these definitions.

A zero $\mathbf{u}$ of A is called *minimal* if for no other zero $\mathbf{v}$ of A, the index set of positive entries of $\mathbf{v}$ is a strict subset of the index set of positive entries of $\mathbf{u}$.

A copositive matrix A is called *irreducible* with respect to another copositive matrix C if for every $\delta > 0$, we have $A - \delta C \notin \mathcal{COP}^n$, and it is called irreducible with respect to a subset $\mathcal{M} \subset \mathcal{COP}^n$ if it is irreducible with respect to all nonzero elements $C \in \mathcal{M}$.

It has been recognised early that the zero set of a copositive matrix is a useful tool in the study of the structure of the cone $\mathcal{COP}^n$ [8,14]. In [3] Baumert considered the possible zero sets of matrices in $\mathcal{COP}^5$. He provided a partial classification of the zero sets of matrices $A \in \mathcal{COP}^5$ which are irreducible with respect to the cone $\mathcal{N}^5$. In [11] this classification was completed, and a necessary and sufficient condition for irreducibility of a copositive matrix $A \in \mathcal{COP}^n$ with respect to the cone $\mathcal{N}^n$ was given in terms of its zero set. In [15], a similar condition in terms of the minimal zero set was given for irreducibility of a copositive matrix $A \in \mathcal{COP}^n$ with respect to the cone $\mathcal{S}^n_+$.

In [9], the facial structure and the extreme rays of the copositive cone $\mathcal{COP}^n$ and its dual, the completely positive cone, have been investigated. It has been shown that not every extreme ray of $\mathcal{COP}^n$ is exposed.

Our main result in this paper is a necessary and sufficient condition on a pair (A, B), where $A \in \mathcal{COP}^n$ and $B \in \mathcal{S}^n$, for the existence of a scalar $\delta > 0$ such that $A + \delta B \in \mathcal{COP}^n$. For fixed A, the set of all such matrices $B \in \mathcal{S}^n$ forms a convex cone $\mathcal{K}^A$, which is referred to as the *cone of feasible directions* [18]. We express this cone in terms of the zeros of A and their supports.

The obtained description of the cone $\mathcal{K}^A$ is a powerful tool. It will allow us to compute the minimal face and the minimal exposed face of A. In particular, we obtain a simple test of extremality of A, which amounts to checking the rank of a certain matrix constructed from the minimal zeros of A. The necessary and sufficient conditions for the irreducibility of A with respect to a nonnegative matrix $C \in \mathcal{N}^n$ or a positive semi-definite matrix $C \in \mathcal{S}^n_+$, which have been given in [11] and [15], respectively, are generalized to the case of arbitrary matrices $C \in \mathcal{COP}^n$. The conditions in [11] and [15] follow as particular cases.

The remainder of the paper is structured as follows. In the next section we provide necessary definitions and notations, and in the following section we collect some results from the literature and provide some preliminary results. In Section 4 we provide our main result, the description of the cone $\mathcal{K}^A$ of feasible directions of $\mathcal{COP}^n$ at A. We also compute its closure, the *tangent cone* $\mathrm{cl}(\mathcal{K}^A)$, and the *tangent space* $\mathrm{cl}(\mathcal{K}^A) \cap -\mathrm{cl}(\mathcal{K}^A)$ [18]. In Section 5 we deduce the descriptions of the minimal face and the minimal exposed face of a copositive matrix. In Section 6 we consider irreducibility of a copositive matrix with respect to another arbitrary copositive matrix. Finally, we give a summary in the last section.

2. Notations

We shall denote vectors with bold lower-case letters and matrices with upper-case letters. Individual entries of a vector $\mathbf{u}$ and a matrix A will be denoted by u_i and a_{ij} respectively. For a matrix A and a vector $\mathbf{u}$ of compatible dimension, the i-th element of the matrix-vector product $A\mathbf{u}$ will be denoted by $(A\mathbf{u})_i$. Inequalities $\mathbf{u} \geq \mathbf{0}$ on vectors will be meant element-wise, where we denote by $\mathbf{0} = (0, \dots, 0)^T$ the all-zeros vector. Similarly we denote by $\mathbf{1} = (1, \dots, 1)^T$ the all-ones vector. We further let $\mathbf{e}_i$ be the unit vector with i-th entry equal to one and all other entries equal to zero. Let $\Delta^n = \{\mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{1}^T\mathbf{x} = 1\}$ be the standard simplex in $\mathbb{R}^n$. For a subset $\mathcal{I} \subset \{1, \dots, n\}$ we denote by $A_\mathcal{I}$ the principal submatrix of A whose elements have row and column indices in $\mathcal{I}$, i.e. $A_\mathcal{I} = (a_{ij})_{i,j\in\mathcal{I}} \in \mathcal{S}^{|\mathcal{I}|}$. Similarly for a vector $\mathbf{u} \in \mathbb{R}^n$ we define the subvector $\mathbf{u}_\mathcal{I} = (u_i)_{i\in\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|}$.