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Second-order Riesz transforms associated with magnetic Schrödinger operators

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ABSTRACT

Let $\mathcal{A} = -(\frac{1}{i}\nabla - \mathbf{a})^2 + V$ be a magnetic Schrödinger operator on $\mathbb{R}^n$, where $\mathbf{a} \in L^2_{loc}(\mathbb{R}^n)^n$ and $0 \le V \in L^1_{loc}(\mathbb{R}^n)$. We show that $L^p(\mathbb{R}^n)$ boundedness of $L_j L_k \mathcal{A}^{-1}$ and $V\mathcal{A}^{-1}$ for some interval of p automatically implies boundedness of the same operators and their commutators on $L^p_w(\mathbb{R}^n)$ for certain Muckenhoupt weights w, and on the Musielak–Orlicz Hardy type spaces.

Contents

1. Introduction

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) : \mathbb{R}^n \to \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$ satisfy

$$\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)^n \qquad \text{and} \qquad 0 \le V \in L^1_{\text{loc}}(\mathbb{R}^n). \tag{1.1}$$

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Consider the magnetic Schrödinger operator:

$$\mathcal{A}(\mathbf{a}, V) := \sum_{j=1}^{n} \big(\frac{1}{i}\partial_j - a_j\big)^2 + V, \qquad \text{on} \quad \mathbb{R}^n, \quad n \geq 2, \tag{1.2}$$

where $\mathbf{a}$ is the magnetic potential and V is the electric potential and $\partial_j = \dfrac{\partial}{\partial x_j}$. Set $L_j := \frac{1}{i}\partial_j - a_j$ for $1 \leq j \leq n$.

The boundedness of the Riesz transforms related to the magnetic Schrödinger operator (1.2) on $L^p(\mathbb{R}^n)$ has attracted much interest in harmonic analysis and has been studied intensively by many mathematicians [1,3,4,13,24,25]. Let us give a brief account of this direction of research.

(a) Under assumption (1.1) it was shown independently by [13] and [26] that the first order Riesz transforms $V^{1/2}\mathcal{A}^{-1/2}$ and $L_k\mathcal{A}^{-1/2}$, with $k = 1, \ldots, n$, are bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq 2$.
(b) With stronger conditions, namely that V belongs to the Muckenhoupt class $\mathbb{A}_\infty$, the authors in [1,24] obtained boundedness for $p > 2$ in the non-magnetic case $\mathcal{A} = -\Delta + V$ (which happens when $\mathbf{a} = \vec{0}$). This was later extended to magnetic operators in [3,4].
(c) The second order Riesz transforms $L_jL_k\mathcal{A}^{-1}$ and $V\mathcal{A}^{-1}$ were considered by Shen in [25], and further studied in [3,4].

Once boundedness on $L^p(\mathbb{R}^n)$ is established it is natural to consider other function spaces. Examples include the weighted spaces $L^p_w(\mathbb{R}^n)$ with Muckenhoupt weights [5,6], and the Hardy spaces $H^p(\mathbb{R}^n)$ and their generalizations [7,9,10,20,30].

In this article we are interested in extending some of these results for the *second order* Riesz transforms. Our aim is to give general statements about boundedness in other function spaces once the $L^p(\mathbb{R}^n)$ boundedness is known.

In several of these works (for example [5,9,20]) the approach to study the Riesz transforms is through the formulae

$$\mathcal{A}^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t\mathcal{A}} \frac{dt}{\sqrt{t}} \qquad \text{and} \qquad \mathcal{A}^{-1} = \int_0^\infty e^{-t\mathcal{A}}\, dt$$

where $e^{-t\mathcal{A}}$ is the heat semigroup associated to $\mathcal{A}$, with heat kernel $p_t(x, y)$. Thus to study the operators $L_j\mathcal{A}^{-1/2}$ and $L_jL_k\mathcal{A}^{-1}$, estimates on the derivatives $L_jp_t(x, y)$ and $L_jL_kp_t(x, y)$ are a crucial element. Typically to obtain such estimates one requires three ingredients:

(i) boundedness of the corresponding Riesz transform on some $L^{p_0}(\mathbb{R}^n)$;
(ii) Gaussian bounds on the heat kernel $p_t(x, y)$;
(iii) specific properties of V and $\mathbf{a}$.

Our observation in this article is that (i) is sufficient to replace (iii) in such calculations. Indeed this was done for the first derivatives $L_jp_t(x, y)$ in [6], and in the current article we do the same for the second derivatives $L_jL_kp_t(x, y)$ (see Proposition 4.1 below). Of course in practice one still requires (iii) to obtain (i) and is generally speaking a non-trivial task.

The main thrust of our results say that once $L^p(\mathbb{R}^n)$ boundedness holds, then one automatically has weighted and Hardy estimates also. They are stated in Theorems 1.1–1.4 below and will assume one or both of the following conditions. Let $p_0 > 1$.