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Uniform stabilization of wave equation with localized damping and acoustic boundary condition

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ABSTRACT

In this paper we deal with stability of the nonlinear wave equation with acoustic boundary conditions to non-locally reacting boundary.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and connected set with smooth boundary Γ, $n \geq 2$. Let Γ_0 and Γ_1 be closed and disjoint subsets of Γ with positive measure such that $\Gamma = \Gamma_0 \cup \Gamma_1$. This work is devoted to the study of the uniform stabilization of solutions of the following nonlinear problem:

$$u'' - \Delta u + a(x) g_1(u') = 0 \quad \text{in } \Omega \times (0, \infty); \tag{1.1}$$

$$u = 0 \quad \text{on } \Gamma_0 \times (0, \infty); \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} = \delta' \quad \text{on } \Gamma_1 \times (0, \infty); \tag{1.3}$$

$$m\delta'' - c^2 \Delta_\Gamma \delta + g_2(\delta') + r\delta = -\rho_0 u' \ \text{ on } \Gamma_1 \times (0, \infty); \tag{1.4}$$

$$u(x, 0) = u_0(x),\ u'(x, 0) = u_1(x) \quad \text{for } x \in \Omega; \tag{1.5}$$

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$$\delta(x,0) = \delta_0(x) \quad \text{for } x \in \Gamma_1, \tag{1.6}$$

$$\delta'(x,0) = \frac{\partial u_0}{\partial \nu}(x), \ \text{ for } x \in \Gamma_1, \tag{1.7}$$

where $' = \dfrac{\partial}{\partial t}$; $\Delta = \displaystyle\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ and Δ_Γ are the Laplace and the Laplace–Beltrami operators, respectively; ν is the outward normal unit vector on Γ_1; $r : \overline{\Gamma_1} \to \mathbb{R}$ is a non-negative continuous function; $g_1, g_2 : \mathbb{R} \to \mathbb{R}$, $u_0, u_1 : \Omega \to \mathbb{R}$ and $\delta_0 : \Gamma_1 \to \mathbb{R}$ are given functions; m, c and ρ_0 are positive constants; and $a : \Omega \to \mathbb{R}$ (the action function) is such that $a(x) \geq a_0 > 0$ over $\omega \subset \Omega$, where ω is a neighborhood of a boundary portion.

The boundary conditions (1.3)–(1.4) are a generalization of the acoustic boundary conditions which were introduced by Beale and Rosencrans [2] into mathematical analysis literature. When the equation (1.1) has no dissipative term ($a \equiv 0$), $c = 0$ and $g_2(x) = dx$, for $x \in \Gamma_1$ and $d > 0$ constant, Beale [1] proved that there is no uniform rate of decay for the associated energy. Therefore three ways have been considered to ensure stability results:

a) *Negligible boundary structure mass ($m = 0$)*: this technique was initially used by Frota, Cousin and Larkin [11] where the authors proved a stabilization result, when $m = 0$ in (1.4) and no internal or boundary damping term involving the function u was added. Many authors have been using this technique, specially when memory terms are introduced. See [4,16,17,20,26–28,32]. See also Graber [18] where the study involving the relations between negligible and non-negligible boundary structure masses was made.
b) *Damping boundary term in the impenetrability equation*: dissipative boundary terms have been used to solve problems involving wave equation with Neumann boundary conditions, see, for example, [22–24]. The strategy consists in introducing to the equation (1.3) appropriate (dissipative) term involving the trace of the function u'. See [18,19,32] where the authors used this strategy with acoustic boundary conditions. We emphasize the work of Graber and Said-Houari [19] where the authors studied the relation damping/source terms.
c) *Internal damping acting over all the domain*: this consists in putting damping terms, which are effective in all domain Ω, in the wave equation, as in [12,13,30,31]. In [12] the authors studied the problem with a general nonlinear internal damping. This internal damping was weakened in [30], but an assumption involving the size of initial data was necessary. In these cases it was possible to get exponential and polynomial decay rates.

The main purpose of this work is to introduce a fourth way to get stabilization for this class of problems, which is between the extremal cases of Beale (without internal damping term) and (c) cases (damping acting in the whole domain). Precisely, we will show that when

a′) *the boundary structure mass is not negligible*,
b′) *the impenetrability equation does not have damping term* and
c′) *internal damping acts only over a subset of the domain (localized damping)*,

it is possible to get uniform decay rates of the energy of the nonlinear problem (1.1)–(1.7). See Fig. 1.

Wave equation with internal localized damping has been extensively studied, see [3,5,6,9,7,21,29,33] and references therein. The characteristic of these problems is that the damping is imposed only over a domain portion. We emphasize the works [3,5,6,9,7] where the authors explored geometric aspects of the domain.

To the best of our knowledge in all papers considering wave equation with acoustic boundary conditions the damping terms are effective in the whole domain. Therefore, the goal of this work is to weaken the damping effect involving the function u, i.e., we prove a uniform stabilization result when the damping acts