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Behaviour in time of solutions to a class of fourth order evolution equations

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ABSTRACT

We consider some initial-boundary value problems for a class of nonlinear parabolic equations of the fourth order, whose solution $u(x,t)$ may or may not blow up in finite time. Under suitable conditions on data, a lower bound for t^* is derived, where $[0,t^*)$ is the time interval of existence of $u(x,t)$. Under appropriate assumptions on the data, a criterion which ensures that u cannot exist for all time is given, and an upper bound for t^* is derived. Some extensions for a class of nonlinear fourth order parabolic systems are indicated.

1. Introduction

In this paper we investigate the blow-up phenomenon of solutions to the following nonlinear parabolic problem of fourth order

$$u_t - k_1(t)\Delta u + k_2(t)\Delta\Delta u = k_3(t)u\,|u|^{p-1}, \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

$$u = 0, \quad \frac{\partial u}{\partial n} = 0 \ \text{ or } \quad \Delta u = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain in R^N, with smooth boundary $\partial\Omega$, $\dfrac{\partial u}{\partial n}$ is the outward normal derivative of u on the boundary $\partial\Omega$, $p > 1$, k_i, $i = 1,2,3$, are positive constants or in general positive derivable functions of t.

The evolution equation (1.1) belongs to a class of parabolic equations of 4th-order: we refer to the book of Galaktionov, Mitidieri and Pohozaev [3] for existence/nonexistence, uniqueness/nonuniqueness, global

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asymptotics, blow-up phenomena to solutions of higher order equations not only of parabolic type, but also of hyperbolic type, and for dispersion and Schrödinger equations.

We point out that for particular classes of 4th-order parabolic equations we have interesting models, as for instance the growth of thin surfaces when exposed to molecular beam epitaxy. More precisely, if $u(x,t)$ is a solution of the following equation

$$u_t + \Delta\Delta u - \nabla \cdot (f(|\nabla u|^2)) = g(x),$$

it can either represent the absolute thickness of the film, or the relative surface height, that is, the deviation of the film height at the point x from the mean film thickness at time t. For the details we refer to [4] and the references therein.

Recently Winkler in [12] considered equation

$$u_t = -\Delta\Delta u - \mu\Delta u - \lambda\Delta(|\nabla u|^2) + f(x)$$

when Ω is a bounded convex domain in R^N, under the conditions $\dfrac{\partial u}{\partial n} = \dfrac{\partial \Delta u}{\partial n} = 0$ on the boundary $\partial\Omega$ with bounded initial data, and studied the existence of global weak solutions.

Moreover Escudero, Gazzola and Peral in [1] proved existence and blow-up results for the equation

$$u_t + \Delta\Delta u = det(D^2 u) + \lambda h(x,t),$$

under (1.2) and (1.3) boundary-initial conditions, which models epitaxial growth processes and where the evolution is dictated by the competition between the determinant of the Hessian matrix of the solution and the bilaplacian (see also [2]).

Our aim is to investigate the question of the blow-up for classical solutions of the problem (1.1), (1.2), (1.3) and to determine lower and upper bounds for the blow-up time t^*, where $[0, t^*)$ is the time interval of existence of $u(x,t)$. In Section 2, we restrict our investigation to a domain $\Omega \subset R^N$, $N = 2$ or 3, and show in these cases that the quantity $\int_\Omega (\Delta u)^2 \, dx$ remains bounded on some time interval $(0, T)$, where T may be explicitly computed in terms of the data of problem (1.1), (1.2), (1.3). Clearly this value of T provides a lower bound for blow-up time t^* of u, if blow-up occurs in L^2-norm, since we have

$$\int_\Omega u^2 \, dx \leq \Lambda_1^{-1} \int_\Omega (\Delta u)^2 \, dx,$$

where Λ_1 is the first eigenvalue of the clamped plate problem defined by

$$\Delta\Delta u_1 - \Lambda_1 u_1 = 0, \quad x \in \Omega,$$

under the boundary condition (1.2).

In Section 3 we establish, under appropriate assumptions on the data, that the quantity $\int_\Omega u^2 \, dx$ cannot remain bounded beyond some computable time T_1. This shows that $u(x,t)$ has to blow up at some time $t^* < T_1$.

In Section 4, we extend the results of Section 2 to a class of nonlinear systems of parabolic equations of fourth order. We refer to [8] and [5–7] (and the references therein) for similar investigations on parabolic equations and systems of second order.

Throughout the paper a comma is used to denote differentiation and for brevity we omit the dependence on time, when it is evident.