J. Math. Anal. Appl. 436 (2016) 729–739

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Lifting convex approximation properties from Banach spaces to their dual spaces and the related local reflexivity

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ARTICLE INFO

Article history:
Received 6 October 2015
Available online 23 December 2015
Submitted by Richard M. Aron

Keywords:
Banach spaces
Banach lattices
Convex approximation properties
Extendable local reflexivity of type C
Principle of local reflexivity of type $B \to A$

ABSTRACT

Our main result demonstrates that rather weak forms of the extendable local reflexivity and of the principle of local reflexivity are needed for the lifting of bounded convex approximation properties from Banach spaces to their dual spaces. This provides a unified approach to the lifting of various bounded approximation properties, including, besides the classical ones, the approximation property for pairs of Banach spaces, and the positive approximation property of Banach lattices.

1. Introduction

Let X and Y be Banach spaces, both real or both complex. We denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators acting from X to Y, and its subspaces of weakly compact and finite-rank operators by $\mathcal{W}(X,Y)$ and $\mathcal{F}(X,Y)$, respectively. We write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$, $\mathcal{W}(X)$ for $\mathcal{W}(X,X)$, and $\mathcal{F}(X)$ for $\mathcal{F}(X,X)$.

Let X be a Banach space and let $1 \le \lambda < \infty$. Recall that X has the λ-*bounded approximation property* if for every compact subset K of X and every $\varepsilon > 0$, there exists $S \in \mathcal{F}(X)$ with $\|S\| \le \lambda$ such that $\|Sx - x\| \le \varepsilon$ for all $x \in K$. Replacing $\mathcal{F}(X)$ by a convex subset of $\mathcal{L}(X)$ containing 0 yields a *convex approximation property* (see Section 2 for the definition).

The study of convex approximation properties was recently launched in [16] (they were occasionally introduced already in [15]). This concept includes, besides the classical approximation properties, various important ones, e.g., positive approximation properties of Banach lattices (see, e.g., [17]) and the approxi-

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http://dx.doi.org/10.1016/j.jmaa.2015.12.033
0022-247X/

mation properties of pairs, consisting of a Banach space and its subspace (introduced and studied by Figiel, Johnson, and Pełczyński in the important paper [7]), providing them a unified approach (see [16]).

Starting from the seminal paper [12] by Johnson, Rosenthal, and Zippin, cases when bounded approximation properties can be lifted from a Banach space X to its dual space X^* have been studied, for instance, in [6,8,10,13,16,19,20,23]. By an important result, due to Johnson and Oikhberg [11], such a lifting is possible when X is extendably locally reflexive.

Definition 1.1. A Banach space X is λ-*extendably locally reflexive* (ELR) if for all finite-dimensional subspaces $E \subset X^{**}$ and $F \subset X^*$, and for all $\varepsilon > 0$, there exists $T \in \mathcal{L}(X^{**})$ such that $T(E) \subset X$, $\|T\| \leq \lambda + \varepsilon$, and $x^*(Tx^{**}) = x^{**}(x^*)$ for all $x^{**} \in E$ and $x^* \in F$.

The ELR was discovered by Rosenthal and studied by Johnson, Oikhberg, and Rosenthal in [11] and [18]. The next theorem is proven in [11, Theorem 3.1 (1)]; for its quantized version, see [18, Theorem 3.13].

Theorem 1.2 *(Johnson–Oikhberg). If a Banach space X is λ-extendably locally reflexive and has the μ-bounded approximation property, then X^* has the $\lambda\mu$-bounded approximation property.*

The proof in [11] relies on the principle of local reflexivity (PLR) (see [20, Corollary 3.13] for an alternative proof which does not use the PLR). The PLR was discovered by Lindenstrauss and Rosenthal [14] in 1969. It was improved by Johnson, Rosenthal, and Zippin [12] in 1971. Since then, many new proofs, refinements, and generalizations of the PLR have been given in the literature (see, e.g., [1] and [22] for results and references). For instance, there is a version of the PLR for Banach lattices due to Conroy and Moore [4], and Bernau [2], revisited in [16]. Very recently, a PLR respecting subspaces was established by the first named author [21].

The method of the proof of Theorem 1.2 in [11] seems to suggest that the latter versions of the PLR could be used, respectively, for the lifting of the positive bounded approximation property and of the bounded approximation property of pairs. Since these approximation properties are special cases of the convex approximation property, a question arises about a *unified approach* to lifting results in the framework of convex approximation properties.

The aim of the present paper is to propose the relevant notions and to establish a lifting theorem for convex approximation properties (see Section 2). As immediate consequences (see Section 3), lifting results follow for the bounded approximation property of pairs (Theorem 1.2 being a special case here) and for the bounded positive approximation property. The lifting Theorem 1.2 has a converse, due to Rosenthal (see [11, Theorem 3.1 (2)] and Theorem 4.1). Inspired by this theorem, the final Section 4 presents converses of lifting results from Sections 2 and 3 together with their applications to some more lifting results.

Our notation is rather standard. The identity operator on a Banach space X is denoted by I_X, and the closed unit ball and the unit sphere of X are denoted by B_X and S_X, respectively. The annihilator of a subspace Y in X is denoted by $Y^\perp := \{x^* \in X^* : \ x^*(y) = 0 \ \ \forall y \in Y\}$. The range of an operator $S : X \to Y$ is denoted by $\operatorname{ran} S := \{Sx : x \in X\}$. For a subset A of $\mathcal{L}(X)$, the dual is defined as $A^* := \{S^* : \ S \in A\}$. We write A^{**} instead of $(A^*)^*$. If B is a subset of $\mathcal{L}(X^{**})$, then $A^{**} \circ B := \{S^{**}T : \ S \in A, \ T \in B\}$. We consider Banach lattices over $\mathbb{R}$. If X is a Banach lattice and A is a subset of $\mathcal{L}(X)$, then $A_+ := A \cap \mathcal{L}(X)_+$ denotes the set of positive operators belonging to A.

2. Lifting of the convex bounded approximation property

Let X be a Banach space and let A be an arbitrary subset of $\mathcal{L}(X)$. The space X has the A-*approximation property* if for every compact subset K of X and every $\varepsilon > 0$, there exists $S \in A$ such that $\|Sx - x\| \leq \varepsilon$ for all $x \in K$. Let $1 \leq \lambda < \infty$. The space X has the λ-*bounded A-approximation property* if S can be chosen