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Generalizations of a terminating summation formula of basic hypergeometric series and their applications

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ABSTRACT

We generalize a terminating summation formula to a unilateral nonterminating, and further, a bilateral summation formula by a property of analytic functions. The unilateral one is proved to be a q-analogue of a ${}_4F_3$-summation formula. And, an identity unifying Jacobi's triple product identity and the quintuple product identity is obtained as a special case of the bilateral one.

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1. Introduction

In this paper, we suppose $0 < |q| < 1$ and follow the notations and terminology in [4]. The q-shifted factorials are defined respectively by

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i) \qquad \text{and} \qquad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

for any integer n. Let

$$(a,b,\cdots,c;q)_k = (a;q)_k(b;q)_k\cdots(c;q)_k,$$

where k is any integer or ∞. The $(m+1)$-basic hypergeometric series Φ (see [4, p. 95, Eqs. (3.9.1) and (3.9.2)]) and Ψ are defined respectively by

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$$\Phi\left(\begin{matrix} a_1,\cdots,a_r: c_{1,1},\cdots,c_{1,r_1}:\cdots: c_{m,1},\cdots,c_{m,r_m} \\ b_1,\cdots,b_{r-1}: d_{1,1},\cdots,d_{1,r_1}:\cdots: d_{m,1},\cdots,d_{m,r_m} \end{matrix}; q, q_1,\cdots,q_m; z\right)$$
$$=\sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_{r-1};q)_n}z^n\prod_{j=1}^{m}\frac{(c_{j,1},\cdots,c_{j,r_j};q_j)_n}{(d_{j,1},\cdots,d_{j,r_j};q_j)_n}$$

and

$$\Psi\left(\begin{matrix} a_1,\cdots,a_r: c_{1,1},\cdots,c_{1,r_1}:\cdots: c_{m,1},\cdots,c_{m,r_m} \\ b_1,\cdots,b_r: d_{1,1},\cdots,d_{1,r_1}:\cdots: d_{m,1},\cdots,d_{m,r_m} \end{matrix}; q, q_1,\cdots,q_m; z\right)$$
$$=\sum_{n=-\infty}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(b_1,\cdots,b_r;q)_n}z^n\prod_{j=1}^{m}\frac{(c_{j,1},\cdots,c_{j,r_j};q_j)_n}{(d_{j,1},\cdots,d_{j,r_j};q_j)_n}.$$

The main results of this paper are Theorems 1.1 and 1.2 below.

Theorem 1.1. *For* $\left|\frac{1}{st}\right|<1$, *there holds*

$$\Phi\left(\begin{matrix} a^2, & aq^2, & -aq^2 & : & s, & t \\ & a, & -a & : & aq/s, & aq/t \end{matrix}; q^2, q; -\frac{1}{st}\right)$$
$$=\frac{(s+t)}{st}\frac{(aq,-q/s,-q/t,aq/st;q)_\infty}{(-q,aq/s,aq/t,-1/st;q)_\infty}.$$

Theorem 1.2. *For* $\left|\frac{a^2}{bst}\right|<1$, *there holds*

$$\Psi\left(\begin{matrix} aq^2, & -aq^2, & b & : & s, & t \\ a, & -a, & a^2q^2/b & : & aq/s, & aq/t \end{matrix}; q^2, q; -\frac{a^2}{bst}\right)$$
$$=\frac{a(s+t)}{(a+1)st}\frac{(q,q/a,aq,aq/st,-a/b;q)_\infty(a^2q^2/bs^2,a^2q^2/bt^2;q^2)_\infty}{(q/s,q/t,aq/s,aq/t,-a^2/bst;q)_\infty(a^2q^2/b,q^2/b;q^2)_\infty}.$$

For miscellaneous summation formulas of basic hypergeometric series, the readers can consult Gasper and Rahman [4].

This paper is organized as follows.

In Section 2, we will firstly prove Theorem 1.1 from a terminating summation formula in [4], and then, Theorem 1.2 will be deduced from Theorem 1.1. In both the proofs, a method similar to that in [5] and [2] is used, where Ramanujan's ${}_1\psi_1$ and Bailey's ${}_6\psi_6$ summation formulas were proved respectively.

In Section 3, we will prove that Theorem 1.1 is a q-analogue of a ${}_4F_3$-summation formula in Andrews, Askey and Roy's book [1].

In Section 4, special cases of Theorem 1.2 will be considered. We will prove that Theorem 1.2 is a generalization of Jacobi's triple product identity and the quintuple product identity.

In the following, LHS (or RHS) means the left (or right) hand side of a certain equality and **N** denotes the set of nonnegative integers.

2. Proofs of Theorems 1.1 and 1.2

The lemma (see, for example, [6, p. 90, Thm. 1.2]) below is the foundation of our proofs in this section.

Lemma 2.1. *Let U be a connected open set and f, g be analytic on U. If f and g agree infinitely often near an interior point of U, then we have $f(z)=g(z)$ for all $z\in U$.*

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