J. Math. Anal. Appl. 436 (2016) 748–763

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Sharp bounds for cumulative distribution functions

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ARTICLE INFO

Article history:
Received 16 May 2015
Available online 23 December 2015
Submitted by M.J. Schlosser

Keywords:
Cumulative distribution functions
Bounds
Marcum functions
Incomplete beta functions
L'Hôpital's rule

ABSTRACT

Ratios of integrals can be bounded in terms of ratios of integrands under certain monotonicity conditions. This result, related to L'Hôpital's monotone rule, can be used to obtain sharp bounds for cumulative distribution functions. We consider the case of noncentral cumulative gamma and beta distributions. Three different types of sharp bounds for the noncentral gamma distributions (also called Marcum functions) are obtained in terms of modified Bessel functions and one additional type of function: a second modified Bessel function, two error functions or one incomplete gamma function. For the noncentral beta case the bounds are expressed in terms of Kummer functions and one additional Kummer function or an incomplete beta function. These bounds improve previous results with respect to their range of application and/or its sharpness.

1. Introduction

Given two functions $f_i(x)$ integrable in $[a, b]$ and continuous in (a, b) and

$$F_i(x) = \int_a^x f_i(t)dt,\ i = 1, 2, \tag{1}$$

we have that, on account of L'Hôpital's rule, the ratio $F_2(x)/F_1(x)$ has the same limit as $f_2(x)/f_1(x)$ for $x \to a^+$ and the same is true for $(F_2(x) - F(b))/(F_1(x) - F(b))$ as $x \to b^-$. Additionally, as we will see, when $f_2(x)/f_1(x)$ is continuous and monotonic, the ratio becomes a bound for the ratio of integrals. This result is closely related to L'Hôpital monotone rule [2, Theorem 2] (see also [1, Lemma 2.2] and [8, p. 42]), which is a useful result for the study of monotonicity and convexity properties (see for instance [4,21,28]).

These bounds on ratios of integrals can be used for bounding cumulative distribution functions $G_i(x) = F_i(x)/F_i(b)$ when additional information is available, like, for instance, the values of the difference $G_2(x) - G_1(x)$ or a known explicit expression for one of the functions $G_i(x)$. We consider this type of bounds for

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http://dx.doi.org/10.1016/j.jmaa.2015.12.024
0022-247X/

the noncentral gamma and beta distributions, obtaining different families of sharp bounds for different selections of the pair of functions G_i. The noncentral gamma distributions are also called Marcum functions and have as a particular case (central case) the incomplete gamma function ratios. On the other hand, the central case for the beta distribution is the incomplete beta function ratio.

The gamma and beta distributions are classical cumulative distributions appearing in many scientific applications. The noncentral gamma distribution is an important function in radar detection and communications and it is widely used in statistics and probability theory (see for instance [24] and references cited therein). Beta distributions are also important functions in statistics and probability. The computation of these functions, particularly for the noncentral case, is difficult. Only recently accurate algorithms for both the lower and upper tail noncentral gamma distributions (Marcum functions P and Q), valid for a large range of parameters, were developed [9,11]. For the noncentral beta case the situation is worse and the available methods of computation [7] are based on the definition of the lower tail distribution in terms of incomplete gamma functions (Eq. (16)). Probably due to this difficulty, many researchers have been involved in studying their properties and in obtaining bounds and approximations in terms of simpler functions, particularly for the case of the non-central gamma distribution (see, for example, [3,10,15,16,24,27]).

We obtain several types of bounds for the gamma and beta distributions. For the noncentral gamma distribution we obtain three types of bounds in terms of modified Bessel functions alone, in terms of a modified Bessel function plus two error functions or in terms of a Bessel function and an incomplete gamma function. The first type of bounds was recently considered in [24] but using a different approach, while the second type is related to some of the bounds in [3]. For the noncentral beta case the bounds are expressed in terms of Kummer functions alone or in terms of an incomplete beta function and a Kummer function. In addition to the bounds related to L'Hôpital rule, we will see how the recurrence relations satisfied by these functions provide complementary bounds. Both for the gamma and beta cases, the bounds we obtain improve previous results with respect to their range of application and/or its sharpness.

We will also prove some monotonicity properties and bounds for Kummer functions which are needed in the construction of the bounds for the beta function (see Appendix A). Additionally, we will briefly explain how the bounds for the distribution functions are useful for computing the inverse of the distributions (named quantile functions in statistics).

2. Bounds related to L'Hôpital's rule

In this section we consider functions defined by integrals. The same results are valid in general for functions $F_i(x)$ satisfying $F_i(a) = 0$ and which are continuous in $[a, b]$ and differentiable in (a, b); we are not loosing generality by writing these functions as integrals. We assume that $F_i(b) \neq 0$ and we will deal with normalized integrals $G_i(x) = F_i(x)/F_i(b)$ (so that $G_1(b) = G_2(b) = 1$); it is clear how to write the results for F_i from those for G_i.

2.1. Bounds for ratios of integrals

The following theorem can be proved by using Rolle's theorem and Cauchy's mean value theorem.

Theorem 1. *Let*

$$G_i(x) = \int_a^x g_i(t)dt,\ i = 1, 2$$

with $G_1(b) = G_2(b) = 1$, $g_1(x)$ *and* $g_2(x)$ *integrable in* $[a, b]$ *and continuous in* (a, b) *and with* $g_1(x) \neq 0$ *in* (a, b). *Let* $\bar{G}_i(x) = 1 - G_i(x)$.