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Lenses and asymptotic midpoint uniform convexity

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ABSTRACT

We introduce a new isometric geometric property, namely asymptotic midpoint uniform convexity and investigate its connection to similar asymptotic properties.

1. Introduction and some preliminary notions and results

Geometric properties of the unit ball of a Banach space are widely used in applications, from nonlinear analysis to best approximation theory. In 1987 S. Rolewicz [15,16] introduced the following property called by him (β)-property: a given real Banach space $(X, \|\cdot\|)$ with closed unit ball B_X is said to possess the (β)-*property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for each $x \in X$ with $1 < \|x\| < 1 + \delta$ the set $\mathrm{conv}(x \cup B_X) \setminus B_X$ has Kuratowski measure of noncompactness less than ε. Here, as usual, for a set

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$A \subset X$, the symbol $\operatorname{conv}(A)$ has the meaning of convex hull of the set A and the Kuratowski measure of noncompactness of the set A, denoted by $\alpha(A)$, is the infimum of all $\varepsilon > 0$ such that A can be covered by a finite number of sets of diameters less than ε. Equivalently, one says that the norm in X possesses property (β). One of the motivations for the introduction of (β)-property was the fact that if, in the above definition, we replace the Kuratowski measure of noncompactness by the diameter of the corresponding set we obtain a characterization of the well-known property of uniform convexity of the norm in X. Another important point is that property (β) implies that the space is reflexive.

Recently, Lima and Randrianarivony [10] used property (β) in its equivalent form given by Kutzarova [7] to answer a ten-year old question of Bates, Johnson, Lindenstrauss, Preiss and Schechtman [2] about uniform quotients. Independently, Revalski and Zhivkov [14] defined the notion of compact uniform convexity in connection to the study of metric projections. We established in [3] that the compact uniform convexity coincides with property (β) of Rolewicz.

In [8], another isometric characterization of property (β), in terms of certain kind of lenses, was given. Let $y \in B_X$ and define

$$\operatorname{Lens}(y,\sigma) := B[y, 1 - \|y\| + \sigma] \backslash B[0,1], \qquad 0 < \|y\| < 1,\ \sigma \in (0, 2\|y\|),$$

where, as usual, $B[z,r]$, $z \in X$ and $r > 0$, stands for the closed ball centered at z and with radius r.

With this definition in hand we recall the following result.

Theorem 1.1. *(See [8].) Let X be a Banach space. If the norm in X has the (β)-property, then for every $0 < t < 1$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that for every x, $\|x\| = 1$, we have $\alpha(\operatorname{Lens}(tx,\delta)) < \varepsilon$. Conversely, if for some $t \in (0,1)$ we have $\alpha(\operatorname{Lens}(tx,\delta)) \to 0$ as $\delta \to 0$ uniformly in x, $\|x\| = 1$, then X has the (β)-property.*

In [3] we have given a further qualitative version of the latter result. Let us mention that, if in the above theorem we replace the Kuratowski measure of noncompactness by the diameter of the corresponding lenses, we obtain a characterization of uniform convexity of the norm in X (cf. [18,8]).

A different type of lenses, again related to the study of uniform-type properties of the norm, is considered by Laakso [9] and by Tyson and Wu [19]. Let $x \in X$ be such that $\|x\| = 1$ and let $\delta \in (0,1)$. Define

$$\operatorname{lens}(x,\delta) := B[x, 1+\delta] \cap B[-x, 1+\delta].$$

It is proved in [9] (see also [19]) that the condition $\operatorname{diam}(\operatorname{lens}(x,\delta)) \to 0$ as $\delta \to 0$ uniformly in x, $\|x\| = 1$, isometrically characterizes uniform convexity of the norm in X.

It is natural to ask the question what kind of geometric property one obtains if one replaces the diameter of the lenses by their Kuratowski index of noncompactness in Laakso's characterization of uniform convexity. It turns out that the new property is not equivalent to property (β). In fact, it will be shown in this article that this new property is implied by asymptotic uniform convexity and is therefore satisfied by the space l_1. In particular, this new property does not imply property (β) isomorphically nor even reflexivity.

2. Asymptotic midpoint uniform convexity

Let $(X, \|\cdot\|)$ be a real Banach space and let S_X denote its unit sphere. We begin this section by the following characterization:

Theorem 2.1. *Let X be a Banach space. Then the following are equivalent:*

(i) $\alpha(\operatorname{lens}(x,\delta)) \to 0$ *as* $\delta \to 0$ *uniformly in* $x \in S_X$*;*