J. Math. Anal. Appl. 436 (2016) 890–910

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Abstract linear partial differential equations related to size-structured population models with diffusion

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ARTICLE INFO

Article history:
Received 7 April 2015
Available online 21 December 2015
Submitted by E. Saksman

Keywords:
Size-structured populations with diffusion
Characteristic curves
Semigroup
Mild solutions
Dual problems
Weak solutions

ABSTRACT

We study abstract linear partial differential equations in Banach spaces and/or Banach lattices related to size-structured population models with spatial diffusion and their dual problems. We introduce mild solutions through semigroup theory and characteristic method and investigate differentiability of mild solutions. Existence of a unique mild solution is shown. Also, a comparison result is obtained as well as the boundedness of mild solutions is investigated in the Banach lattice setting. Furthermore, we consider the dual problems, and then we introduce weak solutions and establish their uniqueness.

1. Size-structured population models with diffusion

Let us consider a biological population living in a habitat $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $p(s,t,x)$ be the population density of size $s \in [0, s_\dagger]$ at time $t \in [0,T]$ in position $x \in \Omega$, where $s_\dagger \in (0,\infty)$ is the finite maximum size, $T \in (0,\infty)$ is a given time. As usual, the spatial diffusion is represented by Laplacian $k\Delta$ with diffusion coefficient $k > 0$ and we assume the individuals do not move outside of Ω through the boundary $\partial\Omega$. Denote by $g(s,t)$ the growth rate of the individuals of size s and time t. Let $\mu(s,t,x)$ and $\beta(s,t,x)$ be the mortality and reproduction rates, respectively, of size s at time t in position x. Let us denote by $f(s,t,x)$ and $C(t,x)$ the inflows of s-size and zero-size individuals, respectively, from outside of the environment. Put $\Omega_T := (0,T) \times \Omega$, $Q := (0,s_\dagger) \times \Omega$, $\mathcal{Q}_T := (0,s_\dagger) \times (0,T) \times \Omega$ and $\Sigma_T := (0,s_\dagger) \times (0,T) \times \partial\Omega$. Size-structured population models with diffusion are formulated as follows:

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http://dx.doi.org/10.1016/j.jmaa.2015.11.077

$$\left.\begin{aligned}
&\partial_t p + \partial_s(g(s,t)p) = k\Delta p(s,t,x) - \mu(s,t,x)p(s,t,x) + f(s,t,x), \quad \text{in } \mathcal{Q}_T,\\
&g(0,t)p(0,t,x) = C(t,x) + \int_0^{s_\dagger} \beta(s,t,x)p(s,t,x)\,ds, \quad \text{in } \Omega_T,\\
&\frac{\partial p}{\partial \nu}(s,t,x) = 0, \quad \text{on } \Sigma_T,\\
&p(s,0,x) = p_0(s,x), \quad \text{in } Q.
\end{aligned}\right\} \tag{1}$$

We assume that the mortality rate $\mu(s,t,x)$ has the form $\mu(s,t,x) = \mu_0(s,t) + \widetilde{\mu}(s,t,x)$ with the natural mortality rate $\mu_0(s,t)$ independent of position x and the other factor $\widetilde{\mu}(s,t,x)$ depending on position x. The natural mortality $\mu_0(s,t)$ is nonnegative but not assumed bounded while $\widetilde{\mu}(s,t,x)$ is bounded but not assumed nonnegative. We also assume the reproduction rate $\beta(s,t,x)$ is bounded.

Let $\mathcal{A}$ be the realization of Laplacian $k\Delta$ in $L^q(\Omega)$, $q \in (1,\infty)$, with the Neumann boundary condition, that is

$$\begin{aligned}
&D(\mathcal{A}) = \left\{ \phi \in W^{2,q}(\Omega) \,\middle|\, \frac{\partial \phi}{\partial \nu}(x) = 0 \text{ a.e. on } \partial\Omega \right\} \subset L^q(\Omega)\\
&\mathcal{A}\phi = k\Delta\phi \quad \text{for } \phi \in D(\mathcal{A}).
\end{aligned}$$

Recall that $\mathcal{A}$ generates an analytic semigroup $\{\mathcal{T}(t) \mid t \geq 0\}$ in $L^q(\Omega)$ and there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\|\mathcal{T}(t)\phi\|_{L^q(\Omega)} \leq Me^{\omega t}\|\phi\|_{L^q(\Omega)}$ for all $\phi \in L^q(\Omega)$. See e.g. [2,5,13]. Define the bounded linear operators $\mathcal{M}(s,t)$ and $\mathcal{B}(s,t)$ in $L^q(\Omega)$ by

$$[\mathcal{M}(s,t)\phi](x) = \widetilde{\mu}(s,t,x)\phi(x), \quad [\mathcal{B}(s,t)\phi](x) = \beta(s,t,x)\phi(x)$$

for $\phi \in L^q(\Omega)$ and let $[C(t)](x) := C(t,x)$. Then (1) can be transformed to the following problem in $X = L^q(\Omega)$:

$$\left.\begin{aligned}
&\partial_t p + \partial_s(g(s,t)p) = [\mathcal{A} - \mu_0(s,t)I - \mathcal{M}(s,t)]p(s,t) + f(s,t), \quad (s,t) \in \overline{S}_T,\\
&g(0,t)p(0,t) = C(t) + \int_0^{s_\dagger} \mathcal{B}(s,t)p(s,t)\,ds, \quad t \in [0,T],\\
&p(s,0) = p_0(s), \quad s \in [0,s_\dagger],
\end{aligned}\right\} \tag{2}$$

where unknown $p(s,t)$ is an $L^q(\Omega)$-valued function.

Size-structured population models without diffusion have been studied in [1,3,6–11] etc. Webb [15] has studied structured population models with age, size and position in connection with semigroup theory, where the reproduction process is described by individuals of age zero and each size s. Compared with [15], our models are focused rather on size-structure and the reproduction process is described by individuals of size zero. Also, our models are inhomogeneous type with the growth rate depending on size and time while [15] deals with the homogeneous models with the growth rate depending only on size.

We develop the abstract theory of partial differential equations in Banach spaces or/and Banach lattices. This enables us to treat size-structured population models with spatial diffusion in rigorous and unified way.

The paper is organized as follows. Section 2 is devoted to the setting of the problem of abstract partial differential equations in Banach spaces with suitable assumptions. In Section 3, we introduce mild solutions through semigroup theory and characteristic methods, and then we derive some properties of mild solutions. We show the existence of a unique nonnegative mild solution in Section 4, where nonnegative is described by the positive cone in ordered Banach space. We establish a comparison result and the boundedness properties