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An embedding technique for the solution of reaction–diffusion equations on algebraic surfaces with isolated singularities

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ABSTRACT

In this paper we construct a parametrization-free embedding technique for numerically evolving reaction–diffusion PDEs defined on algebraic curves that possess an isolated singularity. In our approach, we first desingularize the curve by appealing to techniques from algebraic geometry. We create a family of smooth curves in higher dimensional space that correspond to the original curve by projection. Following this, we pose the analogous reaction–diffusion PDE on each member of this family and show that the solutions (their projection onto the original domain) approximate the solution of the original problem. Finally, we compute these approximants numerically by applying the Closest Point Method which is an embedding technique for solving PDEs on smooth surfaces of arbitrary dimension or codimension, and is thus suitable for our situation. In addition, we discuss the potential to generalize the techniques presented for higher-dimensional surfaces with multiple singularities.

1. Introduction

In this paper we study the problem of evolving a reaction–diffusion PDE on algebraic surfaces that contain isolated singularities. While there are a range of numerical methods for dealing with PDEs on smooth surfaces, there are very few for dealing with PDEs on singular surfaces. In addition to being an interesting problem in its own right, the construction of such a numerical method might be applied to the approximation of PDEs on smooth but highly curved surfaces such as the simulation of heat flow on filaments of high curvature or the development of cortical maps in which regions of high curvature occur along the sulcal lines.

Our approach will use and extend existing embedding techniques. Within the past decade several numerical embedding methods have been created for solving PDEs that are posed on smooth surfaces. These include level-set methods such as those presented in [12] and [11] as well as the Closest Point Method as

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presented in [19] and [22] and further extended in [16] and [17]. However, the existing embedding techniques are low-order accurate or inconsistent when applied to PDEs on surfaces with singularities. This is a result of the fact that within the class of embedding techniques, a smooth representation of the surface is needed in a narrow computational band surrounding the given curve or surface. When the curvature of the surface increases it becomes more difficult, and computationally expensive, to discretize in such a way that the region of high curvature is accurately captured. Moreover, when the curvature becomes infinite, as is the case with a cusp, classical numerical embedding methods may become inaccurate or fail due to the loss of smoothness in the surface representation.

In our method, we evolve PDEs on singular surfaces by evolving a modified PDE on a regularized version of the surface. The first step here is to resolve the singularity in the underlying domain. To this end we employ a standard procedure from algebraic geometry which is known as "blowing-up" [14,13,23]. We construct the blow-up map in such a way that it produces a one-parameter family of smooth surfaces that approximate the original singular surface. Solving the original reaction–diffusion PDE on the one-parameter family of regularized surfaces yields a family of functions which converges to the solution of the original problem posed on the singular surface.

By this approach, the domain of the PDE will change with the parameter. This is an effect which is disadvantageous for the numerics. Thus, we transform the PDE problems so that they all have the same smooth surface as a domain. This produces a one-parameter family of variable coefficient PDEs with coefficients depending now on the parameter which was introduced by the desingularization procedure. The resulting equations are now ready for the application of an embedding technique using standard uniform grid numerical methods on a smooth domain.

In the construction of our numerical technique, we must be mindful of the fact that the blow-up procedure generally embeds the regularized surface into a higher dimensional space. It is therefore necessary to choose a numerical method that is effective for arbitrary co-dimensional embeddings. The Closest Point Method is one such embedding method, as shown in [17] and [22], and is our method of choice.

The paper is organized as follows. In Section 2, we review the definitions relating to smooth embedded surfaces and surface-intrinsic differential operators. Section 3 introduces a motivating example which concretely demonstrates the method that we construct. The example that we present is that of a simple reaction–diffusion equation posed on the closed planar curve given by $y^2 = x^3 - x^4$ which has an isolated cusp singularity at the origin. We will carry this example through the paper to demonstrate our method. In addition, we construct an analytical solution to the given problem which will be used later to assess our numerical results. In Section 3.3 we describe the regularization of the problem in detail. In particular, we demonstrate the blow-up procedure, construct analytical solutions to the regularized problems, prove the convergence to the solution of the original reaction–diffusion equation, and describe the transformation which prepares the problem for an embedding technique. In Section 3.7, we review the basics of the Closest Point Method and apply it to the regularized problem. Moreover, we give convergence studies that demonstrate the robustness of the method. In Section 4 we work through an example of how to apply our method to a surface in 3D with a cusp singularity. In Section 4.1 we explain how to apply the blow-up technique and regularization method to resolve a given surface singularity. Section 5 discusses the extension of the method to surfaces with multiple singularities and also presents some limitations.

2. Algebraic surfaces and surface intrinsic differentials

This section begins by collecting the definitions relating to smooth embedded surfaces and surface-intrinsic differential operators.

In this paper, we consider real algebraic manifolds, i.e. surfaces $S \subset \mathbb{R}^n$ given implicitly as the solution of a system of $m = \operatorname{codim} S$ polynomial equations over the field $\mathbb{R}$. We write this system as

$$\varphi(\mathbf{x}) = 0 \quad \Leftrightarrow \quad \mathbf{x} \in S \tag{1}$$