J. Math. Anal. Appl. 436 (2016) 944–969

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Stabilization of two coupled wave equations on a compact manifold with boundary

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ARTICLE INFO

Article history:
Received 2 May 2015
Available online 23 December 2015
Submitted by X. Zhang

Keywords:
Coupled wave
Energy decay
Stabilization

ABSTRACT

In this paper we study the indirect stabilization of coupled wave equations by an order one term on a compact manifold with boundary. Only one of the two equations is directly damped by a localized damping term. We prove that the energy of smooth solutions of the system decays polynomially under geometric conditions on both the coupling and the damping regions.

1. Introduction and statement of the results

Let (Ω, g) be a C^∞ compact connected n-dimensional Riemannian manifold with boundary Γ. We denote by Δ the Laplace–Beltrami operator on Ω for the metric g. We take two nonnegative smooth functions a and b on Ω. We consider the stabilization problem for the system of coupled wave equations

$$\begin{cases} \partial_t^2 u - \Delta u + b(x)v + a(x)\partial_t u = 0 & \text{in } \mathbb{R}_+^* \times \Omega \\ \partial_t^2 v - \Delta v + b(x)u = 0 & \text{in } \mathbb{R}_+^* \times \Omega \\ u = v = 0 & \text{on } \mathbb{R}_+^* \times \Gamma \\ (u(0,x), \partial_t u(0,x)) = (u_0, u_1) \text{ and } (v(0,x), \partial_t v(0,x)) = (v_0, v_1) & \text{in } \Omega \end{cases} \tag{1}$$

In this paper, we deal with real solutions, the general case can be treated in the same way. With the system above we associate the energy functional given by

$$\begin{aligned} E_{u,v}(t) = \tfrac{1}{2} \int_\Omega \left(|\nabla u(t,x)|^2 + |\nabla v(t,x)|^2 + |\partial_t u(t,x)|^2 + |\partial_t v(t,x)|^2 \right) dx \\ + \int_\Omega b(x)\, u(t,x)\, v(t,x)\, dx. \end{aligned} \tag{2}$$

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http://dx.doi.org/10.1016/j.jmaa.2015.12.014

We assume that a and b are two nonnegative smooth functions such that,

$$\|b\|_\infty \leq \frac{1-\delta}{\lambda^2} \tag{3}$$

for some $\delta > 0$, where λ is the Poincaré's constant on Ω. Under these assumptions we have

$$\begin{aligned} E_{u,v}(0) &= \tfrac{1}{2}\int_\Omega \left(|\nabla u_0(x)|^2 + |\nabla v_0(x)|^2 + |u_1(x)|^2 + |v_1(x)|^2 \right) dx \\ &\quad + \int_\Omega b(x)\, u_0(x)\, v_0(x)\, dx \\ &\geq \tfrac{\delta}{2}\int_\Omega \left(|\nabla u_0(x)|^2 + |\nabla v_0(x)|^2 + |u_1(x)|^2 + |v_1(x)|^2 \right) dx, \end{aligned} \tag{4}$$

for all $(u_0, v_0, u_1, v_1) \in \mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$.

The linear evolution equation (1) can be rewritten under the form

$$\begin{cases} \frac{d}{dt}U + \mathcal{A}U = 0 \\ U(0) = U_0 \in \mathcal{H} \end{cases} \tag{5}$$

where

$$U = \begin{pmatrix} u \\ v \\ \partial_t u \\ \partial_t v \end{pmatrix}, U_0 = \begin{pmatrix} u_0 \\ v_0 \\ u_1 \\ v_1 \end{pmatrix}$$

and the unbounded operator $\mathcal{A}$ on $\mathcal{H}$ is defined by

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & -Id & 0 \\ 0 & 0 & 0 & -Id \\ -\Delta & b & a & 0 \\ b & -\Delta & 0 & 0 \end{pmatrix}$$

with domain

$$\begin{aligned} D(\mathcal{A}) &= \{U \in \mathcal{H}; \mathcal{A}U \in \mathcal{H}\} \\ &= \left(H_0^1(\Omega) \cap H^2(\Omega)\right) \times \left(H_0^1(\Omega) \cap H^2(\Omega)\right) \times H_0^1(\Omega) \times H_0^1(\Omega). \end{aligned}$$

Under our assumptions ($a \geq 0$, $b \geq 0$ and (3)) and from the linear semi-group theory (see for example [1, Theorem 2.1], [2]), we can infer that for $U_0 \in \mathcal{H}$, the problem (5) admits a unique solution $U \in C^0(\mathbb{R}_+, \mathcal{H})$. Moreover we have the following energy estimate

$$E_{u,v}(t) - E_{u,v}(0) = -\int_0^t \int_\Omega a(x)\, |\partial_t u(s,x)|^2\, dx ds \tag{6}$$

for all $t \geq 0$. In addition, if $U_0 \in D\left(\mathcal{A}^k\right)$ for $k \in \mathbb{N}$, then the solution $U \in \cap_{j=0}^k C^{k-j}\left(\mathbb{R}_+, D\left(\mathcal{A}^j\right)\right)$.

A natural necessary and sufficient condition to obtain controllability for wave equations is to assume that the control set satisfies the Geometric Control Condition (GCC) defined in [4,13]. For a subset ω of Ω and $T > 0$, we shall say that (ω, T) satisfies GCC if every geodesic traveling at speed one in Ω meets ω in a