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Global existence and exponential growth of solution for the logarithmic Boussinesq-type equation

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A R T I C L E I N F O

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A B S T R A C T

In this paper we consider the Boussinesq-type equation associated with logarithmic nonlinear term and initial boundary value conditions. By using potential well method combined with the logarithmic Sobolev inequality, we obtain the existence of global solution. At last we show that the L^2-norm of the solution will grow up as an exponential function as time goes to infinity under some suitable conditions for initial data. The result shows that the polynomial nonlinearity is a critical condition of blow-up in finite time for the solutions of Boussinesq equations.

1. Introduction

In this paper, we study the following initial boundary value problem:

$$v_{tt} - v_{xx} + v_{xxxx} + (v_x log|v_x|^k)_x = 0,\ 0 < x < l,\ t > 0, \tag{1.1}$$

$$v(0,t) = v(l,t) = 0, v_x(0,t) = v_x(l,t) = 0,\ t \geq 0, \tag{1.2}$$

$$v(x,0) = v_0(x), v_t(x,0) = v_1(x),\ x \in \Omega, \tag{1.3}$$

where $k \geq 1, \Omega = (0,l), v(x,t)$ denotes the unknown function, the subscripts x and t denote partial derivatives with respect to x and t, $v_0(x)$ and $v_1(x)$ are given initial value functions and satisfy the boundary condition (1.2).

Equation (1.1) arises from a recent paper by Wazwaz [25], where the author firstly introduced the logarithmic Boussinesq equation (log-BE) in the form

$$u_{tt} + u_{xx} + u_{xxxx} + (u log|u|^k)_{xx} = 0 \tag{1.4}$$

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for analysis of the Gaussian solitary wave solutions based on the so-called logarithmic-KdV (log-KdV) equation defined by

$$v_t + v_{xxx} + (vlog|v|)_{xx} = 0.$$

As indicated in [29,7,18], let $u = v_x$ in (1.4), then (1.4) can be transformed into equation (1.1). It is well known that Boussinesq equation

$$u_{tt} - u_{xx} + bu_{xxxx} = a(u^2)_{xx} \tag{1.5}$$

and improved Boussinesq equation

$$u_{tt} - u_{xx} - bu_{xxtt} = a(u^2)_{xx} \tag{1.6}$$

are very important and famous high order strongly nonlinear mathematical physics model equations, where a, b are constants and $b > 0$. The Boussinesq equations (1.5), (1.6) appeared not only in the study of the dynamics of thin inviscid layers with free surface but also in the study of the nonlinear string, the shape-memory alloys, the propagation of waves in elastic rods and in the continuum limit of lattice dynamics or coupled electrical circuits. These equations also arise in other physical applications such as nonlinear lattice waves, iron sound waves in a plasma, and in vibrations in a nonlinear string. Moreover, it was applied to problems in the percolation of water in porous subsurface strata (see [25] and references therein). The study of Cauchy problem for Boussinesq-type equation has attracted considerable attention from many mathematicians and physicists over the last couple of decades. There are many Boussinesq-type equations, such as the generalized Boussinesq equation (or Bq equation)

$$u_{tt} - u_{xx} + bu_{xxxx} = f(u)_{xx}, \tag{1.7}$$

the improved Boussinesq equation (or IMBq equation)

$$u_{tt} - u_{xx} - u_{xxtt} = g(u)_{xx}, \tag{1.8}$$

its higher order generalizations (or generalized double dispersion equations)

$$u_{tt} - u_{xx} + bu_{xxxx} - u_{xxtt} = f(u)_{xx}, \tag{1.9}$$

and the multi-dimensional version of Boussinesq equation

$$u_{tt} - \Delta u - a\Delta u_{tt} + b\Delta^2 u = \Delta f(u),\ a \geq 0. \tag{1.10}$$

A great deal of efforts has been made to establish the sufficient condition for the existence or nonexistence of global solution of Cauchy problem for Boussinesq type equations with various nonlinear terms, such as power-type nonlinearities $f(s) = \pm a|s|^p,\ f(s) = a|s|^p s,\ p > 1$, or more general nonlinearities satisfying either sign properties or some additional structure growth conditions such as $f'(s) \geq C$ (bounded below) or $f(s) = \sum\limits_{k=1}^{m} a_k|s|^p s$ or $f(s) = e^{ks^2}$. As the literature on the Boussinesq-type equation is too extensive to mention all of them, we only partially refer the reader to [31,24,11,21,16,27,19,1,15,33] and references therein.

The practical, quantitative use of the Boussinesq-type equation and its relatives does not always involve the pure initial-value problem. Instead, IBVP on a finite domain or on the half line often come to the fore. To the authors' best knowledge, however, it is seldom discussed on global solutions of the initial boundary value problem for Boussinesq-type equation in the literature. Bona and Sachs [2] studied the