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Parabolic Littlewood–Paley inequality for a class of time-dependent pseudo-differential operators of arbitrary order, and applications to high-order stochastic PDE ☆

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A B S T R A C T

In this paper we prove a parabolic version of the Littlewood–Paley inequality for a class of time-dependent local and non-local operators of arbitrary order, and as an application we show that this inequality gives a fundamental estimate for an L_p-theory of high-order stochastic partial differential equations.

1. Introduction

The classical Littlewood–Paley inequality says (see [12]) that for any $p \in (1, \infty)$ and $f \in L_p(\mathbf{R}^d)$,

$$\int_{\mathbf{R}^d} \left(\int_0^\infty |\sqrt{-\Delta}\, e^{t\Delta} f|^2 dt \right)^{p/2} dx \leq N(p) \|f\|_p^p, \tag{1.1}$$

where $e^{t\Delta} f(x) := \mathcal{S}_t f = p(t, \cdot) * f(\cdot) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbf{R}^d} f(x-y) e^{\frac{-|y|^2}{4t}} dy$. In [5,8], Krylov proved the following parabolic version: for any $p \in [2, \infty)$, $-\infty \leq a < b \leq \infty$, $f \in L_p((a, b) \times \mathbf{R}^d; H)$,

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$$\left\| \left(\int_a^t |(\sqrt{-\Delta} e^{(t-s)\Delta} f)(s,x)|_H^2 \, ds \right)^{1/2} \right\|_{L_p((a,b)\times\mathbf{R}^d)}^p \leq N(p) \| |f|_H \|_{L_p((a,b)\times\mathbf{R}^d)}^p, \tag{1.2}$$

where H is a Hilbert space. If $f = f(x)$ and $H = \mathbf{R}$ then by (1.2) with $a = 0$ and $b = 2$,

$$\begin{aligned} &\int_{\mathbf{R}^d} [\int_0^1 |\sqrt{-\Delta} e^{s\Delta} f|^2 ds]^{p/2} dx \\ &\leq \int_{\mathbf{R}^d} \int_1^2 \left[\int_0^t |\sqrt{-\Delta} e^{(t-s)\Delta} f|^2 ds \right]^{p/2} dt dx \leq 2N(p) \|f\|_{L_p(\mathbb{R}^d)}^p. \end{aligned}$$

This and the scaling $(\sqrt{-\Delta} \mathcal{S}_t f(c\cdot))(x) = \sqrt{-\Delta}(c\mathcal{S}_{c^2 t} f)(cx)$ yield (1.1). Hence (1.2) is a generalization of (1.1). Note that by putting $K(t,s,x) = \sqrt{-\Delta} p(t-s,x)$, we get $\sqrt{-\Delta} e^{(t-s)\Delta} f = K(t,s,\cdot) * f(s,\cdot)$ and therefore (1.2) becomes

$$\left\| \left(\int_a^t |K(t,s,\cdot) * f(s,\cdot)(x)|_H^2 ds \right)^{1/2} \right\|_{L_p((a,b)\times\mathbf{R}^d)}^p \leq N \| |f|_H \|_{L_p((a,b)\times\mathbf{R}^d)}^p. \tag{1.3}$$

In this article we extend Krylov's results [5,8] and provide sufficient conditions on $K(t,s,x)$ so that inequality (1.3) holds.

The following is a special case of Theorem 2.5, which is our main result.

Theorem 1.1. *Let $K(t,s,x)$ be a function defined on $\mathbf{R}^{d+2}$ satisfying*

$$\sup_{(s,\xi)\in\mathbf{R}^{d+1}} \int_s^\infty |\mathcal{F}[K(t,s,\cdot)](\xi)|^2 dt < \infty.$$

Suppose there exist functions $F_i(t,s,x)$ ($i = 1,2,3$) and constants $\kappa_0, C > 0$ such that for any $s < t$ and $x \in \mathbf{R}^d \setminus \{0\}$,

$$\begin{aligned} \left| D_x K(t,s,x) \right| &\leq C(t-s)^{-(d+1)\kappa_0 - \frac{1}{2}} \left| F_1\left(t,s,(t-s)^{-\kappa_0} x\right) \right|, \\ \left| D_x^2 K(t,s,x) \right| &\leq C(t-s)^{-(d+2)\kappa_0 - \frac{1}{2}} \left(\left| F_2(t,s,(t-s)^{-\kappa_0} x) \right| \wedge 1 \right), \\ \left| \frac{\partial}{\partial t} D_x K(t,s,x) \right| &\leq C(t-s)^{-(d+1)\kappa_0 - \frac{3}{2}} \left(\left| F_3(t,s,(t-s)^{-\kappa_0} x) \right| \wedge 1 \right). \end{aligned}$$

Furthermore, assume that there exist constants $\mu_i > d+2$ such that

$$\sup_{s<t} \int |x|^{\mu_i} |F_i(t,s,x)|^2 dx < \infty, \quad (i = 1,2,3).$$

Then for any $p \geq 2$ and $f \in C_0^\infty(\mathbf{R}^{d+1}; H)$,

$$\left\| \Big(\int_{-\infty}^t \left| K(t,s,\cdot) * f(s,\cdot)(x) \right|_H^2 ds \Big)^{1/2} \right\|_{L_p(\mathbf{R}^{d+1})} \leq N \| |f|_H \|_{L_p(\mathbf{R}^{d+1})}, \tag{1.4}$$

where N is independent of f.