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Integrability spaces for the Fourier transform of a function of bounded variation

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ABSTRACT

New relations between the Fourier transform of a function of bounded variation and the Hilbert transform of its derivative are revealed. After various preceding works of the last 25 years where the behavior of the Fourier transform has been considered on specific subspaces of the space of functions of bounded variation, in this paper such problems are considered on the *whole* space of functions of bounded variation. The widest subspaces of the space of functions of bounded variation are studied for which the cosine and sine Fourier transforms are integrable. The main result of the paper is an asymptotic formula for the sine Fourier transform of an *arbitrary* locally absolutely continuous function of bounded variation. Interrelations of various function spaces are studied, in particular, the sharpness of Hardy's inequality is established and the inequality itself is strengthened in certain cases. A way to extend the obtained results to the radial case is shown.

1. Introduction

It apparently was Bochner who first paid attention to the fact that the Fourier transform for functions of bounded variation deserves a unified approach. In fact, many properties of the Fourier transform of a monotone function are given in [8], but the same machinery amounts to functions with bounded variation. We cite a 1959 edition but the book was published already in the 30s. However, this approach has not been taken up, though various results within this scope continued to appear in literature. The paper [24] is also from that period and contains certain contributions to the topic. However, most of them concern either Fourier series of a function of bounded variation or trigonometric series with the sequence of coefficients of bounded variation (see, e.g., [45,7,44,25], and various journal publications). In the last two monographs [44] and [25] many applications of such results are indicated. Let us also mention [10].

We are going to study an important line of this topic in which the Fourier transform of a function of bounded variation is integrable. This subject is relatively recent, essential results have been obtained within

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the last 25 years. There are various arguments to justify our interest in this piece. First of all, and this goes back to the mentioned [8], this gives us an opportunity to understand the Fourier transform in a relatively convenient, improper sense. Second, many of the results obtained in the past and in this paper, have their analogs, or prototypes, in the developed theory of the integrability of trigonometric series, a topic that was born about a hundred years ago. Among numerous results from this theory (see, e.g., [25, Ch. 3, 3.3]) those due to Telyakovskii (the so-called Boas–Telyakovskii condition, see, e.g., [42]) are best matched to the problems under consideration. Further, the spaces of integrability considered till recently are of interest by themselves and have applications in other areas of analysis (see, e.g., [9,17,28]). In a recent survey paper [33] integrability of the Fourier transform is considered in the context of belonging to Wiener's algebra and its relations to the multipliers theory and comparison of operators. Many of integrability conditions originated from the noteworthy result of Trigub [43] on the asymptotic behavior of the Fourier transform of a convex function. However, the most advanced onward integrability results are related to the real Hardy space (see [29,16,25]). This gives rise to an additional argument. We will see in what follows that the well-known extension of Hardy's inequality (see, e.g., [18, (7.24)])

$$\int_{\mathbb{R}} \frac{|\widehat{g}(x)|}{|x|}\,dx \lesssim \|g\|_{H^1(\mathbb{R})} \tag{1}$$

plays crucial role in these considerations. Proposition 4 below asserts that if the integral on the left-hand side of (1) is finite, then g is the derivative of a function of bounded variation f which is locally absolutely continuous (for short, LAC), $\lim\limits_{|t|\to\infty} f(t) = 0$ and its Fourier transform is integrable. This shows that the study of analytic properties of the real Hardy space is unavoidably concerned with integrability properties for the Fourier transform of functions of bounded variation.

Here and in what follows we use the notations "$\lesssim$" and "$\gtrsim$" as abbreviations for "$\le C$" and "$\ge C$", with C being an absolute positive constant.

Let us summarize what is done in this paper. We establish new conditions for the integrability of the Fourier transform in terms of belonging of the derivative of the considered function to certain function space. For the cosine Fourier transform, belonging to one of the considered subspaces of the space of functions of bounded variation ensures only integrability. For the sine Fourier transform, in most cases an asymptotic formula of the form

$$\widehat{f}_s(x) = \int_0^\infty f(t)\sin xt\,dt = \frac{1}{x} f\left(\frac{\pi}{2x}\right) + F(x)$$

is obtained, with F integrable. We study in detail asymptotic behavior of the sine Fourier transform of an *arbitrary* function of bounded variation. The relation is of the asymptotic form

$$\widehat{f}_s(x) = \frac{1}{x} f\left(\frac{\pi}{2x}\right) + \textit{one more term} + F(x).$$

This result can be considered as the main result of the paper; that additional leading term is a key to the full generality within the scope of all functions of bounded variation. By these considerations, we add new spaces to the known scales of integrability spaces. We thoroughly study the relations between them. For instance, proving that the embedding between some of them is proper, we establish the sharpness of Hardy's inequality (1), apparently for the first time in the literature. Further, we find a new scale of nested spaces, intermediate between the known ones. More precisely, we prove that an analog of the known A_q scale forms a class of intermediate spaces between two Hardy type spaces. This is the first step in finding other such scales, more convenient in applications. Returning to analysis of the newly introduced spaces, we realize