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Asymptotics of Racah polynomials with varying parameters

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Within the Askey scheme of hypergeometric orthogonal polynomials, Racah polynomials stay on the top of the hierarchy and they generalize all of the discrete hypergeometric orthogonal polynomials. In this paper, we investigate asymptotic behaviors of Racah polynomials with varying parameters when the polynomial degree tends to infinity. Using the difference equation technique developed in our earlier papers, we obtain an asymptotic formula in the outer region via ratio asymptotics and then derive asymptotic formulas in the oscillatory region via a matching method. Our asymptotic formulas are explicitly given in terms of the polynomial degree, variable and parameters, using elementary functions such as logarithmic, exponential and rational functions. By taking limits, our results also yield asymptotic formulas for orthogonal polynomials in the lower hierarchy of the Askey scheme such as Hahn polynomials and Krawtchouk polynomials.

1. Introduction

The Racah polynomials are named after Racah, because the orthogonal relation is equivalent to that of Racah coefficients or 6-j symbols; see [1]. In [11], Wilson defined the Racah polynomials in terms of a ${}_4F_3$ hypergeometric function. Let $\lambda(x) := x(x+\gamma+\delta+1)$ and N be a nonnegative integer. Define

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) := {}_4F_3\left(\begin{matrix} -n, & n+\alpha+\beta+1, & -x, & x+\gamma+\delta+1 \\ & \alpha+1, & \beta+\delta+1, & \gamma+1 \end{matrix}\middle|\, 1\right), \tag{1.1}$$

where $n = 0, \cdots, N$ and one of the three equalities is satisfied: $\alpha+1 = -N$ or $\beta+\delta+1 = -N$ or $\gamma+1 = -N$. Set

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$$A_n := -\frac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\beta+\delta+1)(n+\gamma+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},$$
$$C_n := -\frac{n(n+\alpha+\beta-\gamma)(n+\alpha-\delta)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}. \tag{1.2}$$

The Racah polynomials (1.1) satisfy the recurrence relation [7, (9.2.3)]

$$\lambda(x)R_n(\lambda(x)) = -A_nR_{n+1}(\lambda(x)) + (A_n+C_n)R_n(\lambda(x)) - C_nR_{n-1}(\lambda(x)).$$

This recurrence relation can be normalized as

$$\pi_{n+1}(z) = (z - A_n - C_n)\pi_n(z) - A_{n-1}C_n\pi_{n-1}(z), \quad \pi_0(z) = 1,\ \pi_1(z) = z - A_0, \tag{1.3}$$

where $z = \lambda(x)$ and

$$\pi_n(\lambda(x)) := \frac{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}{(n+\alpha+\beta+1)_n}R_n(\lambda(x)). \tag{1.4}$$

In addition to the variable x and the degree n, the polynomials in (1.1) involve four free parameters. This inevitably makes the problem of deriving their asymptotic formulas much more complicated. The only result that we can find on this topic is that given by Chen, Ismail and Simeonov [2]. Their method starts with the hypergeometric representation in (1.1). By approximating the ratio of two shifted factorials, they obtain several asymptotic formulas in terms of hypergeometric function ${}_3F_2$ or ${}_2F_1$, when the parameters are fixed.

In the present paper, we are interested in the large-n behavior of $\pi_n(z)$ with varying parameters $\alpha, \beta, \gamma, \delta$. More precisely, we set

$$\alpha + 1 = Na,\ \ \beta = Nb,\ \ \gamma + 1 = Nc,\ \ \delta + 1 = Nd, \tag{1.5}$$

where either $a = -1$ or $b + d = -1$ or $c = -1$. For simplicity, we assume $A_n > 0$ and $C_n > 0$. By Favard's theorem, these conditions guarantee that the zeros of $\pi_n(z)$ are all real and simple; see [3, Sections 1.4 and 1.5]. Thus, we require some additional conditions:

1. when $a = -1$, we assume $b, c, d > 0$ and $b > c + 1$;
2. when $b + d = -1$, we assume $a, b, c > 0$ and $a + b + 1 < c$;
3. when $c = -1$, we assume $a, b, d > 0$ and $a + 1 < d$.

Let $n/N = p$ be a fixed number in $(0, 1)$. We shall derive asymptotic formulas for $\pi_n(N^2y)$ as $N \to \infty$.

2. Main results

Define the ratio $w_k(z) := \pi_k(z)/\pi_{k-1}(z)$ for $k = 1, \cdots, n$. From (1.3), it follows that

$$w_{k+1}(z) = z - (A_k + C_k) - \frac{A_{k-1}C_k}{w_k(z)}. \tag{2.1}$$

Recall $p = n/N$. We obtain from (1.2) and (1.5) that

$$\lim_{N\to\infty}\frac{A_n}{N^2} = A(p); \quad \lim_{N\to\infty}\frac{C_n}{N^2} = C(p),$$

where