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Inverse polynomial mappings and interpolation on several intervals ☆

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ABSTRACT

In the present paper we will use the inverse polynomial image method in order to construct optimal nodes of interpolation on unions of disjoint intervals. We will show how this method works on those disjoint intervals which possess so-called T-polynomials, and also prove that the method becomes ineffective in the absence of T-polynomials.

1. Introduction

Lagrange interpolation has been a widely studied classical area of analysis for more than a century. There is a vast literature discussing the optimal choice of nodes for Lagrange interpolation on a single interval. The question of finding good nodes of interpolation on unions of disjoint intervals turned out to be a much harder problem. In the recent paper [5] the authors verified the existence of optimal nodes with Lebesgue constants of order $O(\log n)$ for any pair of intervals, but this was accomplished without providing an explicit construction. In addition, for the case of two symmetric intervals and certain pairs of non-symmetric intervals explicit nodes with order $O(\log n)$ Lebesgue constants were also found. These nodes were constructed by taking inverse quadratic and cubic polynomial images of the classical Chebyshev nodes. The inverse polynomial image method was introduced by Peherstorfer [7] and Totik [10], and was successfully applied for extending various classical polynomial results from an interval to more general domains. In the present paper we shall develop a unified approach to constructing optimal nodes of interpolation on unions of disjoint intervals using the inverse polynomial image method. We will show how this method works on

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those intervals which possess so-called T-polynomials, and also prove that the method becomes ineffective in the absence of T-polynomials.

For any $s \geq 1$ let

$$-1 = a_1 < b_1 < \cdots < a_s < b_s = 1$$

be a finite partition of the interval $[-1, 1]$, and let

$$J_s := \bigcup_{i=1}^{s} [a_i, b_i] \tag{1}$$

be the corresponding set of s pairwise disjoint intervals.

The Lebesgue function of interpolation on J_s for the system of nodes

$$X_n = \{(-1 \leq) x_n < x_{n-1} < \cdots < x_1 (\leq 1)\} \subset J_s \,, \tag{2}$$

is defined as

$$\lambda(X_n, x) := \sum_{k=1}^{n} |\ell_k(x)| \,, \tag{3}$$

where

$$\ell_k(x) := \frac{\omega_n(x)}{\omega_n'(x_k)(x - x_k)}, \quad \omega_n(x) := \prod_{k=1}^{n} (x - x_k) \,.$$

Furthermore, the Lebesgue constant (the norm of the Lagrange operator) is given by

$$\lambda(X_n) := \|\lambda(X_n, x)\|_{J_s} \,, \tag{4}$$

where $\|\cdot\|_K$ is the supremum norm of the function on any compact set K.

By the classical result of Faber $\lambda(X_n) \geq C \log n$ for any system of nodes on $[-1, 1]$, while various systems of nodes on the single interval $[-1, 1]$ are known for which the optimal order $\lambda(X_n) = O(\log n)$ is attained.

In [5], Theorem 1, it was shown that for any set of nodes $X_n \subset J_s$

$$\lambda(X_n) \geq C \log n$$

with some $C > 0$, i.e. the classic result of Faber (the case $s = 1$) holds for any set of disjoint intervals. (In fact, the proof of Theorem 1 in [5] can be easily extended for any compact set of positive Lebesgue measure on the real line.) On the other hand it was verified in [5], Theorem 2 that there exist systems of nodes in J_s for which the Lebesgue constant is of optimal order $O(\log n)$. However, the proof of the latter result is not constructive. In addition, for certain special pairs of two disjoint intervals explicit construction of optimal nodes of interpolation was also given.

In the present paper we will use the inverse polynomial image method for explicit construction of sets of nodes with optimal $O(\log n)$ order of Lebesgue constant. It turned out that a crucial role in these considerations is played by the so-called T-polynomials studied in detail by Franz Peherstorfer [6]. Let us give the corresponding definition. Denote by Π_m the set of algebraic polynomials of degree at most m. Now recall that a polynomial $p_m \in \Pi_m$ is called the *Chebyshev polynomial on* J_s if $\|p_m\|_{J_s} = 1$ and its leading coefficient is maximal among all polynomials of degree at most m having norm 1 on J_s. This polynomial is known to be unique. Clearly, $J_s \subset p_m^{-1}([-1, 1])$ where