J. Math. Anal. Appl. 436 (2016) 1196–1213

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Remarks on rates of convergence of powers of contractions

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ARTICLE INFO

Article history:
Received 24 January 2015
Available online 21 December 2015
Submitted by J. Bonet

Keywords:
Numerical range
Ritt operators
Quasi-sectorial operators
Rate of convergence of powers
Resolvent estimates
Contractions on L_2

ABSTRACT

We prove that if the numerical range of a Hilbert space contraction T is in a certain closed convex set of the unit disk which touches the unit circle only at 1, then $\|T^n(I-T)\| = \mathcal{O}(1/n^\beta)$ with $\beta \in [\frac{1}{2}, 1)$. For normal contractions the condition is also necessary. Another sufficient condition for $\beta = \frac{1}{2}$, necessary for T normal, is that the numerical range of T be in a disk $\{z : |z-\delta| \leq 1-\delta\}$ for some $\delta \in (0,1)$. As a consequence of results of Seifert, we obtain that a power-bounded T on a Hilbert space satisfies $\|T^n(I-T)\| = \mathcal{O}(1/n^\beta)$ with $\beta \in (0,1]$ if and only if $\sup_{1<|\lambda|<2} |\lambda - 1|^{1/\beta}\|R(\lambda,T)\| < \infty$. When T is a contraction on L_2 satisfying the numerical range condition, it is shown that $T^n f/n^{1-\beta}$ converges to 0 a.e. with a maximal inequality, for every $f \in L_2$. An example shows that in general a positive contraction T on L_2 may have an $f \geq 0$ with $\limsup T^n f/\log n\sqrt{n} = \infty$ a.e.

1. Introduction

Let T be a power-bounded operator on a complex Banach space X. The Katznelson–Tzafriri theorem [22] says that $\|T^n(I-T)\| \to 0$ if and only if the peripheral spectrum $\sigma(T) \cap \mathbb{T}$ is at most the point 1. Moreover, [22, Theorem 5] shows that, when $\sigma(T) \cap \mathbb{T} \subset \{1\}$, also $\|T^n(I-T)^\gamma\| \to 0$ for every $\gamma \in (0,1)$ (where $(I-T)^\gamma = I - \sum_{k=1}^\infty a_k T^k$, with $\{a_k\}_{k\geq 1}$ the coefficients of $(1-t)^\gamma = 1 - \sum_{k=1}^\infty a_k t^k$ for $t \in [-1,1]$, which satisfy $a_k > 0$ and $\sum_{k=1}^\infty a_k = 1$).

The purpose of this paper is to study, for a contraction T on a Hilbert space, the rates of convergence $\|T^n(I-T)\| = \mathcal{O}(1/n^\beta)$, $\beta \in (0,1)$, using spectral and resolvent conditions.

We start by recalling some results relevant to our study. Nagy and Zemánek [33] and Lyubich [30] proved that for an operator T on a Banach space,

$$\sup_n(\|T^n\| + n\|T^n(I-T)\|) < \infty$$

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[1] Research of the first author was partly supported by Israel Science Foundation grant 01/12.

http://dx.doi.org/10.1016/j.jmaa.2015.11.069
0022-247X/

if and only if T satisfies the Ritt resolvent condition [38]

$$\sup_{|\lambda|>1} \|(\lambda - 1)R(\lambda, T)\| < \infty. \tag{1}$$

Komatsu [23] had proved that if T is a bounded operator satisfying the estimate (1) also inside the unit disk, outside a sector with vertex 1 and angle less than π, then T is power-bounded and $\|T^n(I-T)\| = \mathcal{O}(1/n)$. The proof of one direction in [33] is based on a lemma which shows that (1) implies Komatsu's assumptions; the same "sectorial extension" is proved also in [30] (with an optimal sector). Coulhon and Saloffe-Coste [11, Proposition 2] proved (their assumption that $X = L_p$ is not used) that if T is power-bounded and $\|T^n(I-T)\| = \mathcal{O}(1/n)$, then $\|T^n(I-T)^\gamma\| = \mathcal{O}(1/n^\gamma)$ for every $\gamma \in (0,1)$; see also [9, Proposition 6.1].

It follows from Nevanlinna's work [35, Theorem 9] that if T is power-bounded and satisfies, for some $\alpha \in [1,2)$,

$$\sup_{1<|\lambda|<2} |\lambda - 1|^\alpha \|R(\lambda, T)\| < \infty, \tag{2}$$

then $\|T^n(I-T)\| = \mathcal{O}(1/n^{(2-\alpha)/\alpha})$. (The case $\alpha = 1$ is Ritt's condition.)

Dungey [15] obtained several characterizations of the property $\|T^n(I-T)\| = \mathcal{O}(1/\sqrt{n})$, and in [14] he gave several sufficient conditions for a contraction T on a Hilbert space to satisfy this estimate.

Léka [29] has recently constructed, for any $\beta \in (\frac{1}{2}, 1)$, a contraction T in a complex Hilbert space with $\sigma(T) = \{1\}$ and $\|T^n(I-T)\| = \mathcal{O}(1/n^\beta)$. Earlier, Nevanlinna [34, Example 4.5.2] has constructed contractions on $C[0,1]$ with the above rates (but with larger spectra), and Paulauskas [36, Theorem 6] showed how to obtain normal contractions on a (separable) Hilbert space with the above rates.

Cachia and Zagrebnov [7] called a contraction T on a complex Hilbert space *quasi-sectorial* if its numerical range $W(T) := \{\langle Tf, f\rangle : \|f\| = 1\}$ is included in a Stolz region (the closed convex hull of the point 1 and a disk centered at 0 with radius less than 1). They proved [7, Lemma 3.1] that if T is quasi-sectorial, then $\|T^n(I-T)\| = \mathcal{O}(1/n)$; see also [9, Proposition 2.3].

Paulauskas [36] defined generalized quasi-sectorial contractions by the inclusion of their numerical ranges in a certain convex subset of the closed unit disk, larger than a Stolz region (see definition below), and proved that $\|T^n(I-T)\| = \mathcal{O}(1/n^\beta)$ for an appropriate $\beta \in (\frac{1}{2}, 1)$. We offer here a different proof, which under the assumptions of [36] yields a better (larger) value of β as a function of the parameters.

2. A limit theorem for generalized quasi-sectorial contractions

We start this section by defining certain convex subsets of the closed unit disk. The geometric construction of a Stolz region is by taking a circle of radius $r < 1$ centered at 0 and drawing two tangent line segments from the point 1 to this circle. Paulauskas [36] suggests a similar construction, but replacing the tangent line segments by arcs of a *tangent* "parabola-like" curve $x = 1 - b|y|^\alpha$, $1 < \alpha < 2$, $b > 0$, or $\alpha = 2$ and $b > \frac{1}{2}$ (with $|y| \le |y_0| < 1$); we call such a curve a *quasi-parabola*. We denote the obtained convex set by $D(\alpha, b)$, and call it a *quasi-Stolz set*. For a drawing see [36, p. 2078]. The actual construction of $D(\alpha, b)$ is by starting with the parameters α and b, and finding the radius of the corresponding circle; see Lemma 10 of [36]. Whenever we refer to a quasi-Stolz set $D(\alpha, b)$, it is implied that $1 < \alpha \le 2$. An operator with numerical range contained in a quasi-Stolz set is called in [36] *generalized quasi-sectorial*. Note that the numerical radius $w(T) := \sup\{|\langle Tf, f\rangle| : \|f\| = 1\}$ of a generalized quasi-sectorial T is at most 1, so necessarily T is power-bounded with $\sup_n \|T^n\| \le 2$ [42]. Note that curves of the form $x = 1 - b|y|^\alpha$ with $\alpha > 2$ and $b > 0$ are outside the unit disk in a neighborhood of $(1, 0)$, so cannot be used.