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On regularity for the axisymmetric Navier–Stokes equations

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A B S T R A C T

In this paper, we establish a regularity criterion for the Navier–Stokes system with axisymmetric initial data. It is proved that if the local axisymmetric smooth solution u satisfies $\|u^\theta 1_{r\le\varsigma}\|_{L^\alpha((0,T^*);L^\beta)} < \infty$ for any given $\varsigma > 0$, with $\frac{2}{\alpha}+\frac{3}{\beta}=1$, $4 \le \beta \le 6$, then the strong solution keeps smoothness up to time T^*.

1. Introduction

We study the following classic 3D incompressible Navier–Stokes equations in the whole space:

$$\begin{cases} \partial_t u + (u\cdot\nabla u)u + \nabla p = \nu\Delta u, \\ \nabla\cdot u = 0, \\ u(x,t=0) = u_0, \end{cases} \tag{1.1}$$

where $u(x,t) \in \mathbb{R}^3$ and $p(x,t) \in \mathbb{R}$ denote the unknowns, velocity and pressure respectively, while ν denotes the viscous coefficient of the system.

A lot of works have been devoted to study the above system, but global well-posedness for (1.1) with arbitrary large initial data is still a challenging open problem (see [2,4,9]).

Here, we are concerned with (1.1) with axisymmetric initial data. If u_0 is axisymmetric in system (1.1), then the solution $u(x,t)$ of system (1.1) is also axisymmetric [14,8]. So, it is convenient to write $u(x,t)$ as the following form:

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$$u(x,t) = u^r(r,z,t)e_r + u^\theta(r,z,t)e_\theta + u^z(r,z,t)e_z,$$

where e_r, e_θ and e_z are the standard orthonormal unit vectors in cylindrical coordinate system

$$e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0) = (\cos\theta, \sin\theta, 0),$$
$$e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0) = (-\sin\theta, \cos\theta, 0),$$
$$e_z = (0,0,1),$$

with $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$.

By direct computation, it is easy to have the following relations.

$$\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_z)^T = \partial_r e_r + \frac{\partial_\theta}{r} e_\theta + \partial_z e_z,$$
$$\Delta = \nabla\cdot\nabla = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2},$$

and

$$\frac{\partial e_r}{\partial\theta} = e_\theta, \qquad \frac{\partial e_\theta}{\partial\theta} = -e_r.$$

Accordingly, system (1.1) can be rewritten equivalently as the following form:

$$\begin{cases} \frac{\tilde{D}}{Dt}u^r - \nu(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})u^r - \frac{(u^\theta)^2}{r} + \partial_r p = 0, \\ \frac{\tilde{D}}{Dt}u^\theta - \nu(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})u^\theta + \frac{u^r u^\theta}{r} = 0, \\ \frac{\tilde{D}}{Dt}u^z - \nu(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r)u^z + \partial_z p = 0, \\ u|_{t=0} = u_0^r\cdot e_r + u_0^\theta\cdot e_\theta + u_0^z\cdot e_z, \end{cases} \tag{1.2}$$

where $\frac{\tilde{D}}{Dt}$ denotes the material derivative

$$\frac{\tilde{D}}{Dt} = \partial_t + u^r\partial_r + u^z\partial_z.$$

If $u^\theta = 0$ (so-called without swirl), Ukhovskii and Yudovich [14] (see also [8]) proved the existence of generalized solutions, uniqueness and regularity. When $u^\theta \neq 0$ (with swirl), it is very complicated and difficult. For recent progress, one can find results on regularity criteria or global existence with small initial data in [8,10–12,1,8,15]. The question of the regularity criteria to the axisymmetric Navier–Stokes equations with swirl in dependence on the θ-component of the velocity for the first time was studied in [10] and [11]; the former deals with suitable weak solutions and the latter with the Cauchy problem. An important result was proved in [12], where it was shown that if

$$\|u^\theta\|_{L^\alpha((0,T);L^\beta)} < \infty, \text{ with } \frac{2}{\alpha} + \frac{3}{\beta} < 1, \beta > 4,$$

then the solution is smooth up to time T.

The fact that the only singularity may lay on the z-axis follows from [10], so very recently in [15], the following regularity criterion was established:

$$\|u^\theta 1_{r\leq\varsigma}\|_{L^\alpha((0,T);L^\beta)} < \infty, \text{ with } \frac{2}{\alpha} + \frac{3}{\beta} = 1, \text{ with } (\alpha,\beta) = (4,6), \tag{1.3}$$

where $\varsigma > 0$ is given.