J. Math. Anal. Appl. 436 (2016) 1284–1296

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Inverse problems for delay differential equations using the Collage Theorem

Michael Yodzis *, Herb Kunze

Department of Mathematics and Statistics, University of Guelph, Canada

ARTICLE INFO

Article history:
Received 11 February 2015
Available online 21 December 2015
Submitted by R. Popovych

Keywords:
Delay ODEs
Inverse problems
Parameter estimation
Collage Theorem

ABSTRACT

We present a theoretical framework to solve inverse problems for systems of delayed ordinary differential equations (delay ODEs) that allows us to estimate the values of unknown parameters, such as coefficients and time delays, using available time series data. This work builds on similar results for non-delayed ODEs, inspired by the Collage Theorem. We discuss technical details related to the implementation of the method, including the use of non-convex optimization to recover unknown delay values. The performance of the method is demonstrated using simulated and noised datasets to recover parameters in models applied to human health. These include an additive delay model for the population dynamics of malaria transmission, and a distributed delay model for the homeostasis of glucose and insulin in the bloodstream.

1. Introduction

Delayed ordinary differential equations (delay ODEs) are used to model continuous dynamic processes which involve a time lag. In a biological or human health model for example, a lag may result when processes that occur over different time-scales are coupled together. A general delay ODE initial value problem (IVP) is

$$\begin{cases} x'(t) = f_{\boldsymbol{\lambda}}(t, x_t), & t \geq t_0 \\ x(t) = x_{\mathcal{I}}(t), & t_0 - \tau^* \leq t \leq t_0, \end{cases} \tag{1}$$

where $f_{\boldsymbol{\lambda}}$ depends on a finite vector $\boldsymbol{\lambda}$ of parameters, τ^* is the maximum delay, $x_{\mathcal{I}}(t)$ is the initial history function, and the offset notation x_t captures both additive and distributed delays, as discussed in Section 2.1. For each choice of $\boldsymbol{\lambda} \in \Lambda$, Λ a chosen space of parameters, suppose that (1) has solution $\bar{x}_{\boldsymbol{\lambda}}$. In a practical setting, a model's parameters may include coefficients and/or time delays with values that are difficult to observe and measure directly. If observational time series data is available for each of the model

* Corresponding author.
E-mail addresses: myodzis@uoguelph.ca (M. Yodzis), hkunze@uoguelph.ca (H. Kunze).

http://dx.doi.org/10.1016/j.jmaa.2015.11.070
0022-247X/

variables, the problem of estimating the unknown parameter values may be formulated as a delay ODEs inverse problem as follows.

Inverse Problem 1. Let (X, d_X) be an appropriate metric space with $x_{\text{target}} \in X$ and $\bar{x}_{\boldsymbol{\lambda}} \in X$ for all $\boldsymbol{\lambda} \in \Lambda$. Let $\epsilon > 0$ be given. Find parameters $\boldsymbol{\lambda} \in \Lambda$ so that the true approximation error $d_X(x_{\text{target}}, \bar{x}_{\boldsymbol{\lambda}}) < \epsilon$.

Practically, suppose that $x_{\text{target}}(t)$ is a continuous target function, which may be the interpolation of observational data. Then we aim to find parameters so that the delay ODE admits the target function $x_{\text{target}}(t)$ as an approximate solution.

The solution approach we develop in this paper is based on recasting (1) as a fixed point equation for a contractive map T. This follows similar results for non-delayed ODEs [3,8–11] that are built on ideas from fractal imaging [2,10]. In the practical problem above, we seek to approximate a target solution by the fixed point $x_{\boldsymbol{\lambda}}$ of $T_{\boldsymbol{\lambda}}$ (with contraction factor $c_{\boldsymbol{\lambda}}$) for some $\boldsymbol{\lambda}$ in $\Lambda \subset \{\boldsymbol{\lambda} \in \mathbb{R}^n : 0 \leq c_{\boldsymbol{\lambda}} \leq c < 1\}$. The main difference between our approach and that based on Tikhonov regularization [6] is that the constraint $\boldsymbol{\lambda} \in \Lambda$ guarantees that $T_{\boldsymbol{\lambda}}$ is a contraction and, therefore, replaces the effect of the regularization term in the Tikhonov approach.

2. Theory

2.1. Delay ODEs

Ordinary differential equations are used to model continuous dynamic processes in the natural world. To represent a lag or asynchronous process, one can introduce a time delay to the equations under consideration. An additive delay term such as $x(t - \tau_i)$ represents a process that takes a fixed time τ_i to complete, while a distributed delay term like $\int_{t-\sigma_i}^{t} x(s)\,ds$ may represent a cumulative process, or a process that has some uncertainty in its time of completion [14]. Note that each distributed delay can be formulated more generally in terms of a kernel function $k_i(s)$ that acts as a multiplier on the integrand, $\int_{t-\sigma_i}^{t} k_i(s)x(s)\,ds$. However, our analysis and examples focus on the case that all distributed delays have $k_i(s) = 1$. If there are multiple delay terms, say n additive and m distributed, then we define the largest delay

$$\tau^* := \max\left(\tau_1, \ldots, \tau_n, \sigma_1, \ldots, \sigma_m\right). \tag{2}$$

At any time $t \geq t_0$ the state of $x'(t)$ depends on values of $x(t)$ from up to τ^* units previous, so the delay ODE requires an initial history function $x_{\mathcal{I}}(t)$ to be prescribed on an interval at least from $t_0 - \tau^*$ to t_0. In the examples treated in this paper, we consider delay ODE initial value problems of the form

$$\begin{cases} x'(t) = F_{\boldsymbol{\lambda}}\Big(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_n), \\ \qquad\qquad \int_{t-\sigma_1}^{t} x(s)\,ds, \ldots, \int_{t-\sigma_m}^{t} x(s)\,ds\Big), & t \geq t_0 \\ x(t) = x_{\mathcal{I}}(t), & t_0 - \tau^* \leq t \leq t_0, \end{cases} \tag{3}$$

where the parameters $\boldsymbol{\lambda}$ may include vector field coefficients $\mu_1, \ldots, \mu_i$ and time delays $\tau_1, \ldots \tau_n, \sigma_1, \ldots, \sigma_m$. It can be helpful to introduce the delay notation

$$x_t(\theta) := x(t + \theta), \quad -\tau^* \leq \theta \leq 0, \tag{4}$$

where x_t is the current state of the DE, and θ is the offset from the current time t. Using this notation, we may write