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On stabilization of small solutions in the nonlinear Dirac equation with a trapping potential

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A R T I C L E I N F O

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A B S T R A C T

We consider a Dirac operator with short range potential and with eigenvalues. We add a nonlinear term and we show that the small standing waves of the corresponding nonlinear Dirac equation (NLD) are attractors for small solutions of the NLD. This extends to the NLD results already known for the Nonlinear Schrödinger Equation (NLS).

1. Introduction

We consider

$$\begin{cases} \mathrm{i}u_t - Hu + g(u\overline{u})\beta u = 0, \text{ with } (t,x) \in \mathbb{R}\times\mathbb{R}^3, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where for $\mathscr{M} > 0$ we have for a potential $V(x)$,

$$H = D_{\mathscr{M}} + V, \tag{1.2}$$

where $D_{\mathscr{M}} = -\mathrm{i}\sum_{j=1}^{3}\alpha_j\partial_{x_j} + \mathscr{M}\beta$, with for $j = 1,2,3$,

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \beta = \begin{pmatrix} I_{\mathbb{C}^2} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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The unknown u is $\mathbb{C}^4$-valued. Given two vectors of $\mathbb{C}^4$, $uv := u \cdot v$ is the inner product in $\mathbb{C}^4$, v^* is the complex conjugate, $u \cdot v^*$ is the hermitian product in $\mathbb{C}^4$, which we write as $uv^* = u \cdot v^*$. We set $\bar{u} := \beta u^*$, so that $u\bar{u} = u \cdot \beta u^*$.

We introduce the Japanese bracket $\langle x \rangle := \sqrt{1 + |x|^2}$ and the spaces defined by the following norms:

$L^{p,s}(\mathbb{R}^3, \mathbb{C}^4)$ defined with $\|u\|_{L^{p,s}(\mathbb{R}^3,\mathbb{C}^4)} := \|\langle x \rangle^s u\|_{L^p(\mathbb{R}^3,\mathbb{C}^4)}$;

$H^k(\mathbb{R}^3, \mathbb{C}^4)$ defined with $\|u\|_{H^k(\mathbb{R}^3,\mathbb{C}^4)} := \|\langle x \rangle^k \mathcal{F}(u)\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}$, where $\mathcal{F}$ is the classical Fourier transform (see for instance [25]);

$H^{k,s}(\mathbb{R}^3, \mathbb{C}^4)$ defined with $\|u\|_{H^{k,s}(\mathbb{R}^3,\mathbb{C}^4)} := \|\langle x \rangle^s u\|_{H^k(\mathbb{R}^3,\mathbb{C}^4)}$;

$$\Sigma_k := L^{2,k}(\mathbb{R}^3, \mathbb{C}^4) \cap H^k(\mathbb{R}^3, \mathbb{C}^4) \text{ with } \|u\|^2_{\Sigma_k} = \|u\|^2_{H^k(\mathbb{R}^3,\mathbb{C}^4)} + \|u\|^2_{L^{2,k}(\mathbb{R}^3,\mathbb{C}^4)}. \tag{1.3}$$

For $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ consider the bilinear map

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{\mathbb{R}^3} \mathbf{f}(x)\mathbf{g}(x)dx = \int_{\mathbb{R}^3} \mathbf{f}(x) \cdot \mathbf{g}(x)dx. \tag{1.4}$$

We assume the following.

(H1) $g(0) = 0$, $g \in C^\infty(\mathbb{R}, \mathbb{R})$.

(H2) $V \in \mathcal{S}(\mathbb{R}^3, S_4(\mathbb{C}))$ with $S_4(\mathbb{C})$ the set of self-adjoint 4×4 matrices and $\mathcal{S}(\mathbb{R}^3, \mathbb{E})$ the space of Schwartz functions from $\mathbb{R}^3$ to $\mathbb{E}$, with the latter a Banach space on $\mathbb{C}$.

(H3) $\sigma_p(H) = \{e_1 < e_2 < e_3 \cdots < e_n\} \subset (-\mathscr{M}, \mathscr{M})$. Here we assume that all the eigenvalues have multiplicity 1. Each point $\tau = \pm\mathscr{M}$ is neither an eigenvalue nor a resonance (that is, if $(D_{\mathscr{M}} + V)u = \tau u$ with $u \in C^\infty$ and $|u(x)| \leq C|x|^{-1}$ for a fixed C, then $u = 0$).

(H4) There is an $N \in \mathbb{N}$ with $N > (\mathscr{M} + |e_1|)(\min\{e_i - e_j : i > j\})^{-1}$ such that if $\mu \cdot \mathbf{e} := \mu_1 e_1 + \cdots + \mu_n e_n$ then

$$\mu \in \mathbb{Z}^n \text{ with } |\mu| \leq 4N + 6 \ \Rightarrow |\mu \cdot \mathbf{e}| \neq \mathscr{M}, \tag{1.5}$$

$$(\mu - \nu) \cdot \mathbf{e} = 0 \text{ and } |\mu| = |\nu| \leq 2N + 3 \Rightarrow \mu = \nu. \tag{1.6}$$

(H5) Consider the set M_{min} defined in (2.5) and for any $(\mu, \nu) \in M_{min}$ we consider the function $G_{\mu\nu}(x)$ (see the proof of Lemma 5.11 or also later in the introduction the effective hamiltonian), $\widehat{G}_{\mu\nu}(\xi)$ the distorted Fourier transform associated to H (see (5.64) and Appendix A for more details) of $G_{\mu\nu}(x)$ and the sphere $S_{\mu\nu} = \{\xi \in \mathbb{R}^3 : |\xi|^2 + \mathscr{M}^2 = |(\nu - \mu) \cdot \mathbf{e}|^2\}$. Then we assume that for any $(\mu, \nu) \in M_{min}$ the restriction of $\widehat{G}_{\mu\nu}$ on the sphere $S_{\mu\nu}$ is $\widehat{G}_{\mu\nu}|_{S_{\mu\nu}} \neq 0$.

To each e_j we associate an eigenfunction ϕ_j. We choose them such that $\mathrm{Re}\langle \phi_j, \phi_k^* \rangle = \delta_{jk}$. To each ϕ_j we associate nonlinear bound states.

Proposition 1.1 *(Bound states). Fix $j \in \{1, \cdots, n\}$. Then $\exists a_0 > 0$ such that $\forall z_j \in B_{\mathbb{C}}(0, a_0)$, there is a unique $Q_{jz_j} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4) := \cap_{t \geq 0} \Sigma_t(\mathbb{R}^3, \mathbb{C}^4)$, such that*

$$\begin{aligned} &HQ_{jz_j} + g(Q_{jz_j}\overline{Q}_{jz_j})\beta Q_{jz_j} = E_{jz_j} Q_{jz_j}, \\ &Q_{jz_j} = z_j \phi_j + q_{jz_j}, \ \langle q_{jz_j}, \phi_j^* \rangle = 0, \end{aligned} \tag{1.7}$$

and such that we have for any $r \in \mathbb{N}$: