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Duality gap function in infinite dimensional linear programming ☆

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ABSTRACT

The concept of duality gap function in infinite dimensional linear programming is considered in this paper. Basic properties of the function and two theorems on its behavior are obtained by using duality theorems with interior conditions. As illustrations for the results, we investigate the parametric versions of an example due to D. Gale and parametric linear programs on spaces of continuous functions. The notions of Riemann–Stieltjes integral and function of bounded variation have been shown to be very useful for our investigations.

1. Introduction

A linear program is an optimization problem in which both the objective and the constraint are described by linear functions. It is finite dimensional if both the number of programming variables and that of constraints are finite, and is infinite dimensional if these two numbers are infinite.

A rather complete theory for finite dimensional linear programs has been developed. The simplex method for solving such problems, discovered by G.B. Dantzig in 1947, is one of the most famous algorithms of the 20th century. This method has a wide range of applications in optimization. In the last three decades, interior

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methods have been successfully applied to linear programs to reduce computational time for large-scale problems.

For infinite dimensional programs, a satisfactory theory has not yet been completed in some sense, though there are many interesting results. This is an important area of research, as it has significant applications to continuous transportation, piecewise continuous assignments, time-continuous network-flows, space-continuous flow optimization, optimal design of structures, and so on; see e.g. [1,8,12,15]. This type of problems was first considered by R. Bellman [3] in 1957. His problem was in the context of continuous functions of time and is related to a model of linear optimal control used in production systems. In 1956, R.J. Duffin [5] obtained some foundational results of the theory of infinite dimensional linear programming. Many other authors have further contributed to this theory, including D. Gale, L. Hurwicz, K.O. Kortanek, K.S. Kretschmer, J.M. Borwein, A. Shapiro, C. Zalinescu, etc.

Excellent reviews on duality theorems in infinite dimensional linear programming and some related topics were given by Anderson [1] and by Anderson and Nash [2, pp. 61–63]. In the terminology of Bonnans and Shapiro [4], the problems considered in [2] and in [4, pp. 125–132] are *conic linear optimization problems*. They are more complicated than the *generalized linear programs* [4, pp. 132–145], where the ordering cones are generalized polyhedral convex sets.

Some information about recent achievements in infinite dimensional linear programming in general, and in continuous linear programming in particular, can be found in [18–22] and the references therein. Note that by introducing a simple right-hand-side perturbation to the constraint systems (to obtain a sequence of dual pairs of relaxed problems), Wu has succeeded in proving a strong duality theorem [22, Theorem 5.1] guaranteeing not only the equality of the optimal values of the primal problem and the dual problem, but also the existence of solutions for these continuous-time linear problems. Back to the past a little bit, a theory about linear semi-infinite programming, where either the number of constraints or the number of variables is finite, was developed by Goberna and López [6]. We also refer to [7] for several related results about linear and nonlinear semi-infinite programming.

Tight connections of duality theory with sensitivity analysis have been discussed by Gretsky et al. [8] and by Shapiro [17]. In particular, in [8], it has been shown that the value function is subdifferentiable at the primal constraint if and only if there exists an optimal dual solution and there is no duality gap.

Further studies of duality for infinite dimensional linear programs and an investigation about the applications of duality theorems in the directions of the last five chapters of [2] and of the paper by Shapiro [17] would be interesting and of importance.

The aim of this paper is twofold. First, to establish basic properties of the duality gap function, which can serve as a tool for qualitative studies of infinite dimensional linear programs. Second, to analyze the example of Gale (see [2, pp. 42–43]) showing that duality gaps do exist for some dual pairs of infinite dimensional linear programs, in parametric forms. In addition, by using the concepts of Riemann–Stieltjes integral and function of bounded variation, we are able to give a series of illustrative examples for our result on the duality gap function of linear programs on standard dual pairs of Banach spaces.

Some preliminaries are given in the next section. The duality gap function is studied in Section 3, where basic properties of the function and two theorems on its behavior are obtained. To illustrate the obtained results, parametric versions of an example due to D. Gale are analyzed in Section 4, and a series of parametric linear programs on spaces of continuous functions are constructed in Section 5.

2. Preliminaries

We begin by recalling the concept of dual pair of topological vector spaces and some related facts (see Anderson and Nash [2], Robertson and Robertson [16], and the references therein).