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Cauchy problem for a class of quasilinear hyperbolic systems is well posed on a family of Besov spaces

José Ruidival dos Santos Filho[a,*], Rômel da Rosa da Silva[b]

[a] *Universidade Federal de São Carlos, P. O. Box 676, 13565-905, São Carlos SP, Brazil*
[b] *Universidade Tecnológica Federal do Paraná, P. O. Box 571, 85503-390, Pato Branco PR, Brazil*

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ABSTRACT

In this paper we consider the two dimensional Cauchy problem for the quasilinear systems

$$\begin{cases} \partial_t u + a(u)\partial_x u = 0 \\ \qquad u(0,x) = \boldsymbol{u}_0(x), \end{cases}$$

with $u = (u_1, \ldots, u_N)$ and $a(u) = (a_{jk}(u))_{j,k=1}^N$ real $N \times N$ matrix, with entries C^∞, such that the eigenvalues of $a(0)$ are real and distinct, that is, the system is hyperbolic at $u = 0$. We show that a family of Besov spaces, containing the Hölder spaces, near $u = 0$ is continuously preserved by the flow of the above Cauchy Problem.

1. Introduction

In this paper we consider the following quasilinear Cauchy Problem on $\mathbb{R}^2$ given by

$$\begin{cases} \partial_t u + a(u)\partial_x u = 0 \\ \qquad u(0,x) = \boldsymbol{u}_0(x), \end{cases} \tag{1.1}$$

where $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^N$, for $N \geq 1$, under the hypothesis that the system is hyperbolic at $u = 0$. Therefore, if $u = (u_1, u_2, \cdots, u_N)$ and $a(u) = (a_{jk}(u))_{j,k=1}^N$ has entries C^∞ we have that $a(u)$ has real distinct eigenvalues $\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_N(u)$, we denote by $r_1(u), r_2(u), \ldots, r_N(u)$ the corresponding eigenvectors, which depend smoothly on u, in a neighborhood of the origin.

* Corresponding author.
E-mail addresses: santos@dm.ufscar.br (J.R. dos Santos Filho), romelsilva@utfpr.edu.br (R. da Rosa da Silva).

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For the linear case the well known result, called Lax–Mizohata's theorem, shows that hyperbolicity is necessary for the well posedness of the Cauchy problem. For the case when $\mathbb{R}^N$ is replaced by $\mathbb{C}$, J. Hounie and J. R. dos Santos Filho, see [3], under a more restricted notion of well posedness, showed that the equation can be reduced to a semilinear one. Finally, we mention that G. Métivier, in [4], proved a quasilinear version of Lax–Mizohata's theorem. Those results are for analytic, C^∞ and Sobolev spaces.

In [2], Lars Hörmander showed that:

Theorem 1. *For any $\mathbf{u}_0 \in C^\rho$, $1 \le \rho \in \mathbb{Z}$, with bounded derivatives of order $\le \rho$ and those of order ≤ 1 sufficiently small, the Cauchy problem (1.1) has a unique solution $u \in C^\rho$ defined in $[0,T] \times \mathbb{R}$ and it is proved that*

$$T\|\mathbf{u}_0'\|_\infty \le c$$

where c is a constant depending only on a. Moreover, for all multi-indexes α, $0 \le |\alpha| \le \rho$, there are constants $C_{|\alpha|}$, $\tilde{C}_{|\alpha|}$ such that

$$\|\partial^\alpha u(t,\cdot)\|_\infty \le C_{|\alpha|}\|(\mathbf{u}_0)^{(|\alpha|)}\|_\infty \exp(\tilde{C}_{|\alpha|}t).$$

Based in the PhD thesis of the second author, see [6], we prove extensions of the above theorem. Firstly we extend the previous result for the Hölder spaces (which, of course, can be viewed as Besov spaces), in order to make more accessible we prove it with the usual description of the norm of C^ρ instead of considering as the Besov space $B^\rho_{\infty,\infty}$, with the norm described in terms of Littlewood–Paley's decomposition. We have:

Theorem 2. *Let $\mathbf{u}_0 \in C^\rho$, where $1 < \rho$. If $\|\mathbf{u}_0\|_1$ is sufficiently small, then the Cauchy problem (1.1) has a unique solution $u \in C^\rho([0,T] \times \mathbb{R})$, whenever*

$$T\|\mathbf{u}_0'\|_\infty \le c,$$

where c is a constant depending only on a. Moreover, there is a constant C such that

$$\|u\|_\rho \le C\|\mathbf{u}_0\|_\rho. \tag{1.2}$$

Secondly, using paradifferential calculus, see [1] and [5], we extend Theorem 1 for other family of Besov spaces, more precisely we prove:

Theorem 3. *Let $\mathbf{u}_0 \in B^\rho_{\infty,r}$, with $2 < \rho \notin \mathbb{Z}$ and $1 \le r \le \infty$. If $\|\mathbf{u}_0\|_1$ is sufficiently small, then the Cauchy problem (1.1) has a unique solution $u \in C^{[\rho]}([0,T] \times \mathbb{R})$, whenever $T\|\mathbf{u}_0'\|_\infty \le c$ where c is a constant depending only on a. Moreover, $u(t,\cdot) \in B^\rho_{\infty,r}$ and there is a constant C such that*

$$\|u(t,\cdot)\|_{B^\rho_{\infty,r}} \le C\|\mathbf{u}_0\|_{B^\rho_{\infty,r}}, \text{ for all } 0 \le t \le T.$$

In Section 2 we establish notation and recall results that will be useful in the proof of our theorems, in special results of Paradifferential's calculus. In Section 3, we prove the Theorem 2 for $\rho \in (1,2)$. In the following section, we generalize the result of Section 3 for arbitrary ρ, concluding the proof of Theorem 2. Finally, in Section 5, using Paradifferential's calculus, we prove Theorem 3.

2. Preliminaries

In this paper we denote α a multi-index of non-negative integers, for $\rho \in \mathbb{R}_+$ we take $[\rho] = \max\{k \in \mathbb{Z} : k \le \rho\}$ and $C^\rho(\mathbb{R}^n;\mathbb{R}^N)$ is the Hölder space, $0 < \rho \notin \mathbb{Z}$, with the usual norm (denoted by $\|\ \|_\rho$), which