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Pulsating fronts for bistable on average reaction–diffusion equations in a time periodic environment

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ABSTRACT

This paper is devoted to reaction–diffusion equations with bistable nonlinearities depending periodically on time. These equations admit two linearly stable states. However, the reaction terms may not be bistable at every time. These may well be a periodic combination of standard bistable and monostable nonlinearities. We are interested in a particular class of solutions, namely pulsating fronts. We prove the existence of such solutions in the case of small time periods of the nonlinearity and in the case of small perturbations of a nonlinearity for which we know there exist pulsating fronts. We also study uniqueness, monotonicity and stability of pulsating fronts.

1. Introduction and main results

In this paper we investigate equations of the type

$$u_t - u_{xx} = f^T(t,u), \qquad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{1}$$

where

$$f^T(t+T,u) = f^T(t,u), \qquad \forall t \in \mathbb{R}, \quad \forall u \in [0,1],$$

and

$$f^T(t,0) = f^T(t,1) = 0, \qquad \forall t \in \mathbb{R}. \tag{2}$$

Throughout this article, we assume the function $f^T : \mathbb{R} \times [0,1] \to \mathbb{R}$ is of class $\mathcal{C}^1$ with respect to t uniformly for $u \in [0,1]$, and $\mathcal{C}^2$ with respect to x uniformly for $t \in \mathbb{R}$. The main hypotheses imposed on the function f^T are the following

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$$\frac{1}{T}\int_0^T f_u^T(s,0)ds < 0 \quad \text{and} \quad \frac{1}{T}\int_0^T f_u^T(s,1)ds < 0. \tag{3}$$

We say the function f^T is bistable on average if it satisfies hypotheses (2) and (3).

We begin by recalling the definition of monostable and bistable homogeneous nonlinearities. A function $f : [0,1] \to \mathbb{R}$ is said to be monostable if it satisfies $f(0) = f(1) = 0$ and $f > 0$ on $(0,1)$. If in addition to this we have $f(u) \leq f'(0)u$ on $(0,1)$, we say that f is of KPP type. A function $f : [0,1] \to \mathbb{R}$ is called bistable if there exists $\theta \in (0,1)$ such that $f(0) = f(\theta) = f(1) = 0$, $f < 0$ on $(0,\theta)$ and $f > 0$ on $(\theta,1)$.

We give now two examples of bistable on average functions $f^T : \mathbb{R} \times [0,1] \to \mathbb{R}$. The first example is a bistable homogeneous function balanced by a periodic function depending only on time. Namely, if $g : [0,1] \to \mathbb{R}$ is a bistable function and $m : \mathbb{R} \to \mathbb{R}$ is a T-periodic function, then the function f^T defined by $f^T(t,u) = m(t)g(u)$ is bistable on average if and only if the quantity $\frac{1}{T}\int_0^T m(s)ds$ is positive and both $g'(0)$ and $g'(1)$ are negative. The second example of bistable on average function is a combination of a bistable homogeneous function and a monostable homogeneous function, both balanced by periodic functions (with the same period) depending only on time. Namely, if $g_1 : [0,1] \to \mathbb{R}$ is a monostable function, $g_2 : [0,1] \to \mathbb{R}$ is a bistable function, and $m_1, m_2 : \mathbb{R} \to \mathbb{R}$ are two T-periodic functions, then the function f^T defined by $f^T(t,u) = m_1(t)g_1(u) + m_2(t)g_2(u)$ is bistable on average if and only if we have $\mu_1 g_1'(0) + \mu_2 g_2'(0) < 0$ and $\mu_1 g_1'(1) + \mu_2 g_2'(1) < 0$, where $\mu_i := \frac{1}{T}\int_0^T m_i(s)ds$. It is important to note that for a bistable on average function, there can very well exist times t for which the homogeneous function $f^T(t,\cdot)$ is not a bistable function in the sense of homogeneous nonlinearities. Indeed, if we set in the previous case $g_1(u) = u(1-u)$, $g_2(u) = u(1-u)(u-\theta)$ with $0 < \theta < 1$, $m_1(t) = \sin(2\pi t)$ and $m_2(t) = 1 - \sin(2\pi t)$, we can notice that although the function $f^1(t,u) = m_1(t)g_1(u) + m_2(t)g_2(u)$ is bistable on average, the homogeneous function $f^1(1/4,\cdot)$ is of KPP type.

1.1. Context

The study of reaction–diffusion equations began in the 1930's. Fisher [12] and Kolmogorov, Petrovsky and Piskunov [17] were interested in the equation

$$u_t - u_{xx} = f(u), \qquad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{4}$$

with a nonlinearity f of KPP type. They proved existence and uniqueness (up to translation) of planar fronts U_c of speed c, for all speeds $c \geq c^* := 2\sqrt{f'(0)}$, that is, for all $c \geq c^*$, there exists a function u_c satisfying (4) and which can be written $u_c(t,x) = U_c(x-ct)$, with $0 < U_c < 1$, $U_c(-\infty) = 1$ and $U_c(+\infty) = 0$. Numerous articles have been dedicated to the study of existence, uniqueness, stability, and other properties of planar fronts for various nonlinearities, see e.g. [2,11,16,18,24]. In particular, for bistable nonlinearities, there exists a unique (up to translation) planar front $U(x-ct)$ and a unique speed c solution of (4). When the nonlinearity is not homogeneous, there are no planar front solutions of (1) anymore. For equations with coefficients depending on the space variable, Shigesada, Kawasaki and Teramoto [28] defined in 1986 a notion more general than the planar fronts, namely the pulsating fronts. This notion can be extended for time dependent equations as follows.

Definition 1.1. A pulsating front connecting 0 and 1 for equation (1) is a solution $u : \mathbb{R} \times \mathbb{R} \to [0,1]$ such that there exists a real number c and a function $U : \mathbb{R} \times \mathbb{R} \to [0,1]$ verifying

$$\begin{cases} u(t,x) = U(t, x-ct), & \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}, \\ U(\cdot,-\infty) = 1,\ \ U(\cdot,+\infty) = 0, & \text{uniformly on } \mathbb{R}, \\ U(t+T,x) = U(t,x), & \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}. \end{cases}$$